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1st Rank, JEE ADVANCED
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You Asked
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MATHEMATICS today

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CONTENTS

- Success Story 4
- Maths Musing Problem Set - 140 7
- Enrich Your Concepts 8
  Class XI (Series-3)
- Maths Musing - Solutions 16
- You Asked, We Answered 18
- Olympiad Corner 19
- CBSE Board Chapterwise 24
  Class XII (Series-3)
- Concept Boosters (XI) 35
- Concept Boosters (XII) 53
- Mock Test Paper (Series-2) 75
  JEE (Main & Advanced) - 2015

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editorial

Education and Degrees from Universities need Optimization

We have one India and the system of education should be unified. We have a misconception that more the fees one pays, and more the time taken to pass out of the Universities, better is the education. In some cases, it is true but not always. In many states, people work hard so that they can educate the children at least upto the plus 2 stage. Then they learn computers and their use in offices as short – hand and typewriting has been displaced now. They have to come to cities for better salaries. Although in some states, they find good pay and satisfaction.

This is the economic reason for education. State run colleges and universities have to keep this in view. The faster the education is completed, after taking degrees, they can join our national research laboratories as well as specialised centres of research where the scientists really learn science, applied as well as research. But their career is made. No amount of education in universities will really replace the specialisation they have to get from the centres of research. Longer the courses, better the education is a fallacy. Education is a lifelong process. Research performed by professors in colleges is not learnt from the Universities alone, although that is also very essential as otherwise gaps in knowledge could hamper research.

Our educationists have a definite target for the proper orientation of our students. They can play full part in the development of our students. No Indian University will consider adjusting their programs so that students can pass GRE and other exams to go to U.S.A or other countries. Our aim is to excel foreign universities not because we do not like foreign countries. Our race for excellence is to excel ourselves.

Anil Ahlawat
Editor

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My SUCCESS STORY

Chitraang Murdia
1st Rank JEE Advanced

- **Why did you appear for engineering entrance?**
  I was interested in Maths and Science and the best option for me was clearing the engineering entrance exam.
- **What exams you have appeared for and what is your rank in these exams?**
  I scored 335/360 in JEE Mains which is the qualifying stage for JEE Advanced. By the grace of God and with the support of my parents and teachers I topped JEE Advanced with AIR-I.
- **Do you have any other Achievements?**
  I have been selected to represent India in IPhO in Kazakhstan. I have also being selected for all four National Olympiads in Physics, Chemistry, Mathematics and Astronomy.
- **How many hours in a day did you study to prepare for the exams?**
  I studied for a total of 13 hours in a day.
- **On which topic and chapters you lay more stress in each subject?**
  I laid more emphasis on Conic Section and Calculus in Maths, Rotational Dynamics and Electro Magnetic Induction in Physics and Carbonyl Compounds, p-block elements and Electrochemistry in Chemistry.
- **How much time does one require for serious preparation for this exam?**
  6-8 hours of self study are sufficient to prepare for this exam.
- **Any extra coaching?**
  I took coaching from Allen Career Institute and the guidance of my teachers helped me to prepare for this exam.
- **Which subject/topics you were strong / weak at?**
  I had good command on calculus and electromagnetism and I was weak in p-block and conic section.
- **Which books / magazines / tutorials / coaching class you followed?**
  I was a regular reader of your esteemed magazine Mathematics Today and Physics for You. Whenever I found time I also referred to Chemistry Today. I had also joined Allen Career institute.

MTG comes out with a lot of research materials on many occasions. When I wanted to test myself on a topic I turned to Concept Boosters in Mathematics Today which I found to be a blend of all types of problems. While preparing for Mathematics Olympiad I read the articles "*Team Selection Tests by Alok Sir*" and found them challenging. Many of the ideas helped me to become a better problem solver and added to my readiness for IIT.

- **In your words what are the components of an ideal preparation plan?**
  Hardwork and proper analysis of mistakes are essential for an ideal preparation plan.
- **What role did the following play in your success?**
- **(a) Parents (b) Teachers and (c) School**
  (a) My parents gave me full support and motivation in my preparation.
  (b) My teachers guided me properly and solved my queries whenever required.

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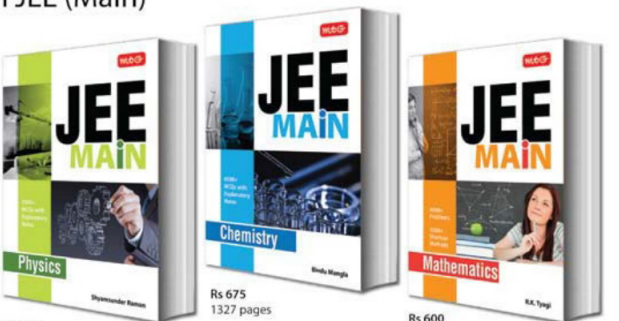

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(c) A proper environment is essential for a healthy competition.

- **Your family background?**
  My father is B.Tech (IIT, Delhi) and my mother is B.A. Eco (Honours). Both of them runs their own business.

- **What mistake you shouldn't have made?**
  I thought on the D-day I was able to keep my cool and things went according to me but I was totally wrong as I made a lot of silly mistakes.

- **How have MTG magazines helped you in your preparation?**
  MTG comes out with a lot of research materials on many occasions. When I wanted to test myself on a topic I turned to Concept Boosters in Mathematics Today which I found to be a blend of all types of problems. While preparing for Mathematics Olympiad, I read the article on "Team Selection Tests by Alok Sir" and found them challenging. Many of the ideas helped me to become a better problem solver and added to my readiness for IIT.

- **Was this your first attempt?**
  Yes.

- **What do you think is the secret of your success?**
  Proper analysis of mistakes and removal of my doubts at the correct time. I worked a lot on the areas I thought that wanted more attention.

- **What advice would you like to give our readers who are PET aspirants?**
  There is no shortcut to success. Hardwork is salvation.

❍❍

IIT Whiz-Kids *Success Mantra*

Pallerla Sai Sandeep Reddy JEE Advanced '13 Topper

"Have a good study plan based on the syllabus. Divide time between subjects based on your interest and difficulties faced. Study topic you are not good at. Spend more time on subjects you are not good.at. Time division is highly important, so divide your time accordingly. While studying, complete concentration is of utmost importance. Do not study continuously. Study for 50-60 minutes and then take a 10 minute break. Interact with your teachers as much as possible.". *– Mathematics Today, August '13*

Arpit Agarwal IIT-JEE 2012 Topper

"To be focussed and manage time well. Sincere hardwork for 2 years is a must I managed time in a way that I could finish my syllabus and revise as many times as I wanted. I think aspirants should manage time intelligently". *– Mathematics Today, June '12*

Prudhvi Tej Immadi IIT-JEE 2011 Topper

"Even if you spend 10 hours reading your books in a day, you might not crack IIT. But if you relax and take your examination with a confident cool mind you have a better chance of cracking it," he said, adding that other social activities are important to keep oneself at peace.
– Mathematics Today, July '11

Anumula Jithendar Reddy IIT-JEE '10 Topper

Keep your cool. Study the subjects with interest and the rest will follow.
– Mathematics Today, July '10

Nitin Jain IIT-JEE & AIEEE 2009 Topper

Proper time management is the key to success in IIT-JEE.
"Plan out yout studies, create a schedule, and stick to it. Do not leave out any topic thinking that it will not come, because chances are it just might."
– Mathematics Today, July '09

Shitikant IIT-JEE 2008 Topper

"After understanding the concepts, it is not a big deal to get a good rank in IIT-JEE." "There's no use of slogging away for 18-19 hours without the right methodology".
"IIT-JEE is not one of the toughest examinations. JEE questions are not difficult. One has to apply one's mind while attmepting questions." He did not burn the midnight oil while preparing for the exam. He never studied over 7-8 hours a day in the run-up to the exam.
– Mathematics Today, July '08

Achin Bansal IIT-JEE 2007 Topper

"I am never tense during the preparation, but I always ensure that if my input is about 10 hours, my output should also about 10 hours".
"In Science it is not cramming but indepth understanding of the subject which pays."
– Mathematics Today, July '07

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material. During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 140

JEE MAIN

1. The locus of a point which is equidistant from the lines $x - y = 0, z = 1$ and $x + y = 0, z = -1$ is
(a) $xy = z$ (b) $xy + z = 0$
(c) $xy = 2z$ (d) $xy + 2z = 0$

2. Let M be the number of positive integer solutions of $x + y + z + u \le 20$. The number of prime factors of M is
(a) 2 (b) 3 (c) 4 (d) 5

3. $M = 54^2 + 53^2 + 52^2 - 51^2 - 50^2 - 49^2 + 48^2 + 47^2 + 46^2 - 45^2 - 44^2 - 43^2 + \ldots - 3^2 - 2^2 - 1^2$. The number of prime factors of M is
(a) 3 (b) 4 (c) 5 (d) 6

4. $P(x)$, a polynomial of degree 3 has maximum at $x = 1$. If $P'(x)$ has minimum at $x = 1$, then $P(x)$ has minimum at $x =$
(a) 0 (b) 1 (c) 2 (d) 3

5. If a, b, c, d and p are real numbers such that $(a + ib)^{(c + id)} = p > 0$, then $(a^2 + b^2)^{(c^2 + d^2)} =$
(a) p^{c^2} (b) p^{d^2}
(c) p^{2c} (d) p^{2d}

JEE ADVANCED

6. The last integer greater than $((5^5 + 1)^{1/5} - 5)^{-1}$ is divisible by
(a) 2 (b) 3
(c) 5 (d) 7

COMPREHENSION

Consider a triangle formed by the points t_1, t_2, t_3 on the parabola $y^2 = 4ax$ such that the orthocentre of the triangle is the focus of the parabola.

7. The locus of the centroid of triangle is a conic of latus rectum
(a) $2a$ (b) $\frac{2a}{3}$ (c) $\frac{4a}{3}$ (d) $\frac{5a}{3}$

8. The locus of the circumcentre of the triangle is a conic of latus rectum
(a) $2a$ (b) $\frac{2a}{3}$ (c) $\frac{4a}{3}$ (d) $\frac{5a}{3}$

INTEGER MATCH

9. In triangle ABC, $a = 4$, $b = 3$ and $c = \frac{\pi}{2}$. The altitude from C on the side AB is CD. A circle is described on CD as diameter. Tangents from A and B to the circle meet at E. The area of triangle ABE is

MATCHING LIST

10. Let C be the coefficient of x^n in the expansion of $(1 + x + x^2)^n$. Column-I gives values of n and Column-II gives the product of digits of C.

Column-I		Column-II	
P.	3	1.	4
Q.	4	2.	5
R.	5	3.	7
S.	6	4.	9

	P	Q	R	S
(a)	1	2	3	4
(b)	2	1	4	3
(c)	3	4	2	1
(d)	4	3	2	1

See Solution set of Maths Musing 139 on page no. 16

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

ENRICH YOUR CONCEPTS ON XI

Trigonometric Functions and Principle of Mathematical Induction

Based on NCERT Pattern

Series-3

TRIGONOMETRIC FUNCTIONS

MEASUREMENT OF AN ANGLE

There are three systems of measuring an angle :

(i) Sexagesimal system or English system :
One degree (1°) is $(1/360)^{th}$ of the complete revolution.
1 right angle = 90°, 1° = 60′, 1′ = 60″

(ii) Centesimal system or French system :
1 right angle = 100 ^{g}, 1^g = 100′, 1′ = 100″

(iii) Circular system : A radian is an angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.
1 right angle = $(\pi/2)$ radians
The length l of an arc PQ of a circle of radius r is given by $l = r\theta$, where θ is the angle subtended by arc PQ at the centre of the circle in terms of radians.

RELATIONSHIP BETWEEN THREE SYSTEMS

2 right angles = 180° = 200 ^{g} = π radians
1 radian = $(180/\pi)$ = 57°16′ (approx.)
1° = $(\pi/180)$ radian = 0.01746 radian (approx.)
Note : Radian measure = $(\pi/180)$ × Degree measure and Degree measure = $(180/\pi)$ × Radian measure

TRIGONOMETRIC FUNCTIONS

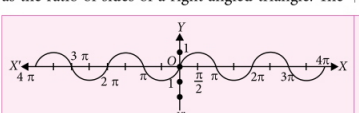
Trigonometric ratios for an acute angle is defined as the ratio of sides of a right angled triangle. The extension of trigonometric ratios to any angle in terms of radian measure are called trigonometric functions.

SIGNS OF TRIGONOMETRIC FUNCTIONS IN FOUR QUADRANTS

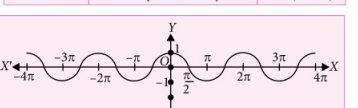
	I	II	III	IV
sin x	+	+	–	–
cos x	+	–	–	+
tan x	+	–	+	–
cosec x	+	+	–	–
sec x	+	–	–	+
cot x	+	–	+	–

DOMAIN AND RANGE OF TRIGONOMETRIC FUNCTIONS

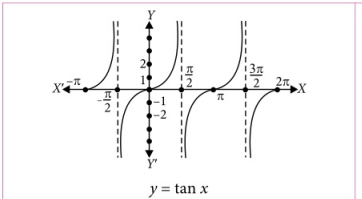
Functions	Domain	Range
sine	R	$[-1, 1]$
cosine	R	$[-1, 1]$
tan	$R-\{(2n+1)\frac{\pi}{2} : n \in Z\}$	R
cot	$R-\{n\pi : n \in Z\}$	R
sec	$R-\{(2n+1)\frac{\pi}{2} : n \in Z\}$	$R-(-1, 1)$
cosec	$R-\{n\pi : n \in Z\}$	$R-(-1, 1)$



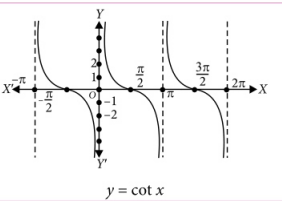
$y = \sin x$



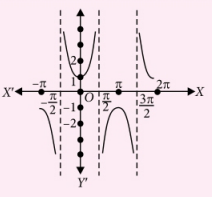
$y = \cos x$



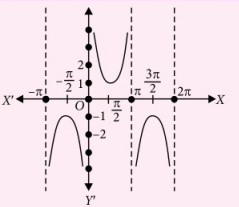
$y = \tan x$



$y = \cot x$



$y = \sec x$



$y = \text{cosec } x$

SINE, COSINE AND TANGENT OF SOME ANGLES LESS THAN 90°

	0°	15°	18°	30°	36°	45°	60°	90°
sine	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$2-\sqrt{3}$	$\frac{\sqrt{25-10\sqrt{5}}}{5}$	$\frac{1}{\sqrt{3}}$	$\sqrt{5-2\sqrt{5}}$	1	$\sqrt{3}$	not defined

ALLIED ANGLES

Two angles are said to be allied iff their sum or their difference is a multiple of 90°. *i.e.* The angle $\frac{n\pi}{2} \pm \theta$ are called allied or related angles and $\theta \pm n \times 360°$ are called co-terminal angles.

Note :

1. *t*-ratios of an angle do not change if it is increased or decreased by a multiple of 360°.

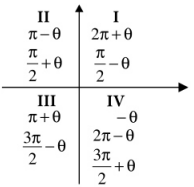
2. In case of $-\theta$, $\pi \pm \theta$, $2\pi \pm \theta$, *t*-ratio remains the same and in case of $\frac{\pi}{2} \pm \theta, \frac{3\pi}{2} \pm \theta$, *t*-ratio changes as follows : sin → cos, cos → sin, tan → cot, cot → tan, sec → cosec, cosec → sec.

3. $\frac{\pi}{2} - \theta$, $2\pi + \theta$ lying in the first quadrant

$\frac{\pi}{2} + \theta$, $\pi - \theta$ lying in the second quadrant

$\pi + \theta, \frac{3\pi}{2} - \theta$ lying in the third quadrant

$\frac{3\pi}{2} + \theta$, $2\pi - \theta$, $-\theta$ lying in the fourth quadrant



FUNCTION OF NEGATIVE ANGLES

Let θ be any angle. Then
sin (−θ) = − sinθ, cos (−θ) = cos θ
tan (−θ) = −tanθ, cot (−θ) = − cot θ
sec (−θ) = secθ, cosec (−θ) = − cosec θ

SOME FORMULAE REGARDING COMPOUND ANGLES

An angle made up of the sum or differences of two or more angles is called a compound angle. The basic results in this direction are called trigonometric identies as given below :

- $\sin(A + B) = \sin A \cos B + \cos A \sin B$
- $\sin(A - B) = \sin A \cos B - \cos A \sin B$
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$
- $\cos(A - B) = \cos A \cos B + \sin A \sin B$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
- $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$
- $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$
- $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$
- $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1 + \tan^2 A}$
- $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$

  $= 2\cos^2 A - 1 = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$
- $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$
- $\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\cos A - \cos B = 2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{B-A}{2}\right)$
- $\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$
- $\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$
- $2\sin A\cos B = \sin(A + B) + \sin(A - B)$
- $2\cos A \sin B = \sin(A + B) - \sin(A - B)$
- $2\cos A\cos B = \cos(A + B) + \cos(A - B)$
- $2\sin A\sin B = \cos(A - B) - \cos(A + B)$
- $\sin\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$ $\begin{cases} +\text{if } \dfrac{A}{2} \text{ lies in I or II quadrants} \\ -\text{if } \dfrac{A}{2} \text{ lies in III or IV quadrants} \end{cases}$
- $\cos\dfrac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$ $\begin{cases} +\text{if } \dfrac{A}{2} \text{ lies in I or IV quadrants} \\ -\text{if } \dfrac{A}{2} \text{ lies in II or III quadrants} \end{cases}$
- $\tan\dfrac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$ $\begin{cases} +\text{if } \dfrac{A}{2} \text{ lies in I or III quadrants} \\ -\text{if } \dfrac{A}{2} \text{ lies in II or IV quadrants} \end{cases}$

TRIGONOMETRIC EQUATIONS

Definition : Equation involving one or more than one trigonometric ratios of unknown angles is called trigonometric equations.

SOLUTION OF TRIGONOMETRIC EQUATION

A value of unknown angle which satisfies the given trigonometric equation is called a solution of the trigonometric equation.

TYPES OF SOLUTION

I. Principal Solution : The solution of a trigonometric equation lying between $[0, 2\pi]$ is called principal solution.

Principal Solutions of Trigonometric Equations

(i) If $\sin x = \sin\alpha$, then principal solution is $x = \alpha$ and $x = \pi - \alpha$

(ii) If $\cos x = \cos\alpha$, then principal solution is $x = \alpha$ and $x = 2\pi - \alpha$

(iii) If $\tan x = \tan\alpha$, then principal solution is $x = \alpha$ and $x = \pi + \alpha$

II. General Solution : The expression involving integer k which gives all solutions of trigonometric equation is called the general solution.

General Solutions of Trigonometric Equations

(i) General Solution of $\sin x = 0$ is $x = n\pi$, $n \in Z$

(ii) General solution of $\tan x = 0$ is $x = n\pi$, $n \in Z$.

(iii) General solution of $\cos x = 0$ is $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$.

(iv) General solution of $\cot x = 0$ is $x = (2n + 1)\frac{\pi}{2}$, $n \in Z$

(v) General solution of $\sin x = \sin\alpha$ is $x = n\pi + (-1)^n\alpha$, $n \in Z$

(vi) General solution of $\cos x = \cos\alpha$ is $x = 2n\pi \pm \alpha$, where $n \in Z$.

(vii) General solution of $\tan x = \tan\alpha$ is $x = n\pi \pm \alpha$, $n \in Z$.

PRINCIPLE OF MATHEMATICAL INDUCTION

First principle of mathematical induction

Step I : Actual verification of the proposition for the starting value '*i*'.

Step II : Assuming the proposition is to be true for '*k*', $k \le i$ and then proving that it is true for the value $(k + 1)$ which is the next higher integer.

Step III : Combine the above two steps or let $P(n)$ be a statement involving natural number n.

To prove statement $P(n)$ is true for all natural numbers we use following process :

1. Prove that $P(1)$ is true.
2. Assume $P(k)$ is true.
3. Using (1) and (2), prove that statement is true for $n = k + 1$, *i.e.*, $P(k + 1)$ is true.

Second principle of mathematical induction

Step I : Actual verification of the proposition for the starting value i and $(i + 1)$.

Step II : Assuming the proposition to be true for $(k - 1)$ and then proving that it is true for the value $(k + 1)$, $k \le i + 1$.

Step III : Combine the above two steps, these are used to solve a problem or in second principle of Mathematical Induction following steps are used :

1. Prove that $P(1)$ is true.
2. Assume $P(n)$ is true for all natural numbers such that $2 \le n < k$.
3. Using (1) and (2), prove that $P(k + 1)$ is true.

VERY SHORT ANSWER TYPE (1 MARK)

1. A wheel makes 180 revolutions in one minute. Through how many radians does it turn in one second?

2. A circular wire of radius 3 cm is cut and bent so as to lie along a circle of radius 48 cm. Find the angle subtended by the wire at the centre of the circle.

3. Find the ciruclar measure of the following angles :

(i) 75° **(ii)** $-22°30'$

4. Find the degree measure of $\left(-\frac{\pi}{3}\right)^c$

5. Find x from the equation

$\operatorname{cosec}(90° + \theta) + x\cos\theta\cot(90° + \theta) = \sin(90° + \theta)$.

6. Find the maximum and minimum values of $\sin^2\theta + \cos^4\theta$.

SHORT ANSWER TYPE (4 MARKS)

7. Simplify $\tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{5\pi}{20}\tan\frac{7\pi}{20}\tan\frac{9\pi}{20}$.

8. Prove that $\tan 20° + 4\sin 20° = \sqrt{3}$.

9. Let $P(n)$ be the statement "$n^2 + n$ is an odd integer." Show that if $P(m)$ is true then $P(m + 1)$ is also true.

10. Solve the equation $2\sin^2\theta = 3\cos\theta$ in the interval $0 \le \theta \le 2\pi$

11. Find the general solution of the equation $\sin 2x + \sin 4x + \sin 6x = 0$.

LONG ANSWER TYPE (6 MARKS)

12. Prove by mathematical induction that the sum of the squares of first n natural numbers is $\frac{1}{6}n(n+1)(2n+1)$.

13. By principle of mathematical induction, show that $a^n - b^n$ is divisible by $a + b$, when n is a positive even integer.

14. Solve the equation $\sec x \cos 5x + 1 = 0$, for $0 < x < 2\pi$.

15. Solve $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$ in the interval $0 \le x \le 2\pi$.

SOLUTIONS

1. Number of revolutions made in one second

$$= \frac{180}{60} = 3$$

Also, angle turned when the wheel makes one revolution $= 2\pi$ radians

$\therefore$ Number of radians turned in one second

$$= 3 \times 2\pi = 6\pi.$$

2. Length of the wire $= 2\pi(3 \text{ cm}) = 6\pi$ cm. Let this wire (when made to lie along a circle of radius 48 cm) make an angle θ radians at the centre, then $\theta = \frac{6\pi \text{ cm}}{48 \text{ cm}} = \frac{\pi}{8}$. $\left(\because \theta = \frac{l}{r}\right)$

3. **(i)** Now, $180° = \pi$ radian $\Rightarrow 1° = \frac{\pi}{180}$ radian

$$\Rightarrow \quad 75° = \frac{75\pi}{180}\text{radian} = \frac{5\pi}{12}\text{radian}$$

$\therefore$ Circular measure of an angle of 75° is $\frac{5\pi}{12}$.

(ii) $(-22°30') = \left(-22\frac{1}{2}\right)^\circ = \frac{\pi}{180}\left(-\frac{45}{2}\right)$ radian

$$\left(\because 30' = \left(\frac{30}{60}\right)^\circ = \left(\frac{1}{2}\right)^\circ\right)$$

$$= -\frac{\pi}{8} \text{ radian}$$

4. Since $\pi^c = 180°$, therefore, $1^c = \left(\frac{180}{\pi}\right)^\circ$

$$\Rightarrow \quad \left(-\frac{\pi}{3}\right)^c = \left(\frac{180}{\pi} \times -\left(\frac{\pi}{3}\right)\right)^\circ = -60°.$$

5. The given quation is

$\operatorname{cosec}(90° + \theta) + x\cos\theta\cot(90° + \theta) = \sin(90° + \theta)$

$\Rightarrow \sec\theta + x\cos\theta(-\tan\theta) = \cos\theta$

$\Rightarrow \sec\theta - x\sin\theta = \cos\theta$

$$\Rightarrow \quad x\sin\theta = \sec\theta - \cos\theta = \frac{1}{\cos\theta} - \cos\theta$$

$$\Rightarrow \quad x\sin\theta = \frac{1-\cos^2\theta}{\cos\theta} = \frac{\sin^2\theta}{\cos\theta}$$

$\Rightarrow x = \tan\theta.$

6. $\sin^2\theta + \cos^4\theta = \sin^2\theta + (1 - \sin^2\theta)^2$

$$= \sin^4\theta - \sin^2\theta + 1 = \left(\sin^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$\therefore$ Minimum value $= \frac{3}{4}$ and

maximum value $= \frac{1}{4} + \frac{3}{4} = 1$

7. $\tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{5\pi}{20}\tan\frac{7\pi}{20}\tan\frac{9\pi}{20}$

$$= \tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{\pi}{4}\tan\left(\frac{\pi}{2} - \frac{3\pi}{20}\right)\tan\left(\frac{\pi}{2} - \frac{\pi}{20}\right)$$

$$\left(\because \frac{7\pi}{20} = \frac{10\pi - 3\pi}{20} = \frac{\pi}{2} - \frac{3\pi}{20} \text{ and } \frac{9\pi}{20} = \frac{\pi}{2} - \frac{\pi}{20}\right)$$

$$= \tan\frac{\pi}{20}\tan\frac{3\pi}{20}\tan\frac{\pi}{4}\cot\frac{3\pi}{20}\cot\frac{\pi}{20}$$

$$\left(\because \tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta\right)$$

$$= \left(\tan\frac{\pi}{20}\cot\frac{\pi}{20}\right)\tan\left(\frac{3\pi}{20}\cot\frac{3\pi}{20}\right)\tan\frac{\pi}{4}$$

$= 1 \times 1 \times 1 = 1 \qquad (\because \tan\theta\cos\theta = 1)$

8. We can write, $\tan 20° + 4\sin 20° = \tan 60°$

$\Rightarrow \tan 60° - \tan 20° = 4\sin 20°.$

Now, L.H.S. $= \tan 60° - \tan 20°$

$$= \frac{\sin 60°}{\cos 60°} - \frac{\sin 20°}{\cos 20°}$$

$$= \frac{\sin 60°\cos 20° - \cos 60°\sin 20°}{\cos 60° \cdot \cos 20°}$$

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$$= \frac{\sin(60^\circ - 20^\circ)}{\cos 60^\circ \cos 20^\circ} = \frac{\sin 40^\circ}{\frac{1}{2} \times \cos 20^\circ}$$

$$= \frac{2 \sin 20^\circ \cos 20^\circ}{\frac{1}{2} \cos 20^\circ} = 4 \sin 20^\circ = \text{R.H.S.}$$

Hence, $\tan 60^\circ - \tan 20^\circ = 4 \sin 20^\circ$.

9. Given statement is $P(n)$: "$n^2 + n$ is an odd integer"

Let $P(m)$ be true $\Rightarrow$ $m^2 + m$ is an odd integer ...(i)

Now, we have to show $P(m + 1)$:

"$(m + 1)^2 + (m + 1)$ is an odd integer."

Here, $(m + 1)^2 + (m + 1) = m^2 + 2m + 1 + m + 1$

$= (m^2 + m) + (2m + 2)$

= an odd integer + 2 $(m + 1)$ (using (i))

= an odd integer + an even integer

($\because$ 2 divides 2 $(m + 1)$)

= an odd integer.

Hence, $P(m + 1)$ is true. So, $P(m) \Rightarrow P(m + 1)$.

10. We have, $2 \sin^2 \theta = 3 \cos \theta$

$\Rightarrow$ $2(1 - \cos^2 \theta) = 3 \cos \theta$

$\Rightarrow$ $2 \cos^2 \theta + 3 \cos \theta - 2 = 0$

$\Rightarrow$ $(2 \cos \theta - 1)(\cos \theta + 2) = 0$

$\therefore$ either $2 \cos \theta - 1 = 0$ or $\cos \theta + 2 = 0$

Now, $2 \cos \theta - 1 = 0 \Rightarrow \cos \theta = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$

$\therefore$ $\theta = 2n\pi \pm \pi/3$, $n \in I$

Also, $\cos \theta + 2 = 0 \Rightarrow \cos \theta = -2$ which is not possible as $-1 \le \cos \theta \le 1$

$\therefore$ $\theta = 2n\pi \pm \pi/3$, $n \in I$

But since we require the value of θ in $[0, 2\pi]$ only

$\therefore$ required values are $\pi/3$ and $5\pi/3$.

11. The given equation may be written as

$(\sin 6x + \sin 2x) + \sin 4x = 0$

$$\Rightarrow 2 \sin\left(\frac{6x + 2x}{2}\right)\cos\left(\frac{6x - 2x}{2}\right) + \sin 4x = 0$$

$\Rightarrow$ $2 \sin 4x \cos 2x + \sin 4x = 0$

$\Rightarrow$ $\sin 4x (2 \cos 2x + 1) = 0$

$\Rightarrow$ $\sin 4x = 0$ or $2 \cos 2x + 1 = 0$

$$\Rightarrow \sin 4x = 0 \text{ or } \cos 2x = \frac{1}{2} = -\cos\frac{\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = \cos\frac{2\pi}{3}$$

$$\Rightarrow \sin 4x = 0 \text{ or } \cos 2x = \cos\frac{2\pi}{3}$$

$$\Rightarrow 4x = n\pi \text{ or } 2x = \left(2m\pi \pm \frac{2\pi}{3}\right), \text{ where } m, n \in I$$

$$\Rightarrow x = \frac{n\pi}{4} \text{ or } x = \left(m\pi \pm \frac{\pi}{3}\right), \text{ where } m, n \in I.$$

Hence, the general solution is given by $x = \frac{n\pi}{4}$ or $x = \left(m\pi \pm \frac{\pi}{3}\right)$, where $m, n \in I$.

12. Let $P(n)$ be the given mathematical statement *i.e.*,

$$P(n) : 1^2 + 2^2 + 3^2 + ...+ n^2 = \frac{1}{6} n(n + 1)(2n + 1) \quad ...(1)$$

Putting $n = 1$ in (1), we get

$$P(1) : 1^2 = \frac{1}{6} \cdot 1 \cdot (1 + 1)(2 \cdot 1 + 1)$$

i.e., $P(1) : 1 = \frac{1}{6} \cdot 2 \cdot 3$ or $1 = 1$, which shows that $P(n)$ is true for $n = 1$

Let us assume that $P(n)$ is true for $n = m$; then

$$1^2 + 2^2 + 3^2 + ... + m^2 = \frac{1}{6} m(m + 1)(2m + 1) \quad ...(2)$$

Now, we show that the statement is true for, $n = m + 1$

Now, $1^2 + 2^2 + 3^2 + ... + m^2 + (m + 1)^2$

$= \frac{1}{6} m(m + 1)(2m + 1) + (m + 1)^2$ [using (2)]

$= \frac{1}{6} (m + 1)[m(2m + 1) + 6(m + 1)]$

$= \frac{1}{6} (m + 1)(2m^2 + m + 6m + 6)$

$= \frac{1}{6} (m + 1)(2m^2 + 7m + 6)$

$= \frac{1}{6} (m + 1)(2m^2 + 4m + 3m + 6)$

$= \frac{1}{6} (m + 1)[2m(m + 2) + 3(m + 2)]$

$= \frac{1}{6} (m + 1)(m + 2)(2m + 3)$

$= \frac{1}{6} (m + 1)[(m + 1) + 1][2(m + 1) + 1]$ ...(3)

From (3), it follows that $P(m + 1)$ is true whenever $P(m)$ is true.

Since $P(1)$ is true and $P(m+1)$ is true, hence by the principle of mathematical induction, it follows that $P(n)$ is true for all $n \in N$ *i.e.*, the sum of the squares of first n natural numbers is

$$\frac{1}{6}n(n+1)(2n+1)$$

13. By question, n is an even positive integer; hence, we take $n = 2k$ where $k \in N$, the set of natural numbers. Therefore, we have to show that $(a^{2k} - b^{2k})$ is divisible by $a + b$.

Let $P(k)$ be the statement : $(a^{2k} - b^{2k})$ is divisible by $a + b$.

$\therefore$ $P(1) : a^2 - b^2 = (a - b)(a + b)$ is divisible by $a + b$, which is true ; hence, $P(1)$ is true.

Let $P(m) : (a^{2m} - b^{2m})$ is divisible by $a + b$ be true *i.e.*, $P(m)$ is true.

Since $P(m)$ is true *i.e.*, $(a^{2m} - b^{2m})$ is divisible by $a + b$, hence if $P(m) = (a^{2m} - b^{2m})$ then we can write,

$P(m) = (a + b)\,\varphi(a, b)$,

where $\varphi(a, b)$ is a polynomial in (a, b) ...(1)

Now, $P(m+1) = a^{2(m+1)} - b^{2(m+1)}$

$= a^{2m} \times a^2 - b^{2m} \times b^2$

or, $P(m+1) = [(a + b)\,\varphi(a, b) + b^{2m}]a^2 - b^2 \times b^{2m}$

$[\because\ P(m) = (a^{2m} - b^{2m}) = (a + b)\,\varphi(a, b)$

$\therefore\ a^{2m} = (a + b)\,\varphi(a, b) + b^{2m}]$

or $P(m+1) = a^2(a + b)\,\varphi(a, b) + (a^2 - b^2)b^{2m}$

$= (a + b)[a^2\,\varphi(a, b) + (a - b)b^{2m}]$ ...(1)

From (1), it readily follows that $P(m+1)$ is divisible by $(a + b)$ *i.e.*, $a^{2(m+1)} - b^{2(m+1)}$ is divisible by $(a + b)$.

This implies that $P(m+1)$ is true.

Since $P(1)$ is true and $P(m+1)$ is true whenever $P(m)$ is true, hence by principle of mathematical induction, it follows that $P(k)$ is true for all $k \in N$ *i.e.*, $a^n - b^n$ is divisible by $(a + b)$ when n is an even positive integer.

14. We have $\sec x \cos 5x + 1 = 0$.

$\Rightarrow \sec x \cos 5x = -1\ (\cos x \neq 0)$

$\Rightarrow \cos 5x = -\cos x = \cos(\pi - x)$

$\therefore\ 5x = 2n\pi \pm (\pi - x)$

Taking positive sign,

$5x = 2n\pi + \pi - x = (2n+1)\pi - x$

$\Rightarrow 6x = (2n+1)\pi \Rightarrow x = (2n+1)\pi/6$

Taking negative sign, $5x = 2n\pi - \pi + x$

$\Rightarrow 4x = (2n-1)\pi \Rightarrow x = (2n-1)\pi/4$

But $0 < x < 2\pi$.

$\therefore$ Putting $n = 0, 1, 2, 3, 4, 5$ in $x = (2n+1)\frac{\pi}{6}$, we get

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$ in the given range.

And putting $n = 1, 2, 3, 4$ in $x = (2n-1)\frac{\pi}{4}$, we get $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ in the given range.

Out of these values $x = \frac{\pi}{2}, \frac{3\pi}{2}$ are rejected as $\cos x = 0$ for these values. $\therefore$ Required values of n are $\frac{\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{6}, \frac{5\pi}{4}, \frac{7\pi}{6}, \frac{7\pi}{4}, \frac{11\pi}{6}$

15. $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$.

$\Rightarrow$ $(\sin x + \sin 3x) + \sin 2x$

$= (\cos x + \cos 3x) + \cos 2x$

$\Rightarrow 2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{3x-x}{2}\right) + \sin 2x$

$= 2\cos\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right) + \cos 2x$

$\Rightarrow 2\sin 2x \cos x + \sin 2x = 2\cos 2x \cos x + \cos 2x$

$\Rightarrow \sin 2x\ (1 + 2\cos x) = \cos 2x\ (1 + 2\cos x)$

$\Rightarrow (\sin 2x - \cos 2x)(1 + 2\cos x) = 0$

If $\sin 2x - \cos 2x = 0 \Rightarrow \tan 2x = 1 = \tan\frac{\pi}{4}$

$\Rightarrow 2x = n\pi + \frac{\pi}{4} \Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8}$ ...(i)

If $1 + 2\cos x = 0 \Rightarrow \cos x = -\frac{1}{2} = \cos\frac{2\pi}{3}$

$\Rightarrow x = 2n\pi \pm \frac{2\pi}{3}$ ...(ii)

Now, in the given interval, $0 \le x \le 2\pi$, from (i), we get, for $n = 0, 1, 2, 3$

$\Rightarrow x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

From (ii), we get, for $n = 0, 1$

$\Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\therefore$ Required values of x in the given interval are $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{9\pi}{8}, \frac{4\pi}{3}, \frac{13\pi}{8}$

❑❑

MATHS MUSING

SOLUTION SET-139

1. (d) : 1 to 9 $\Rightarrow$ 9 digits, 10 to 99 $\Rightarrow$ 180 digits

100 to 699 $\Rightarrow$ 1800 digits $\therefore$ 1 to 699 $\Rightarrow$ 1989 digits

Now, 700, 701, 702, 703, 704, 705, 706, 707, 708

2014^{th} digit = $(1989 + 8 \times 3 + 1)^{\text{th}}$ digit = 7, the first digit of 708.

2. (a): $7^{2014} = -(1-50)^{1007}$

$= -[1 - 50350 + 1007 \cdot 503 \cdot 2500$

$- 1007 \cdot 503 \cdot 335 \cdot 125000]$

$= -1 + 350 - 2500 + 5000 = 2849$, digit sum = 23.

3. (c): $\left(\frac{a+b+c+d}{4}\right)^2 \le \frac{a^2+b^2+c^2+d^2}{4}$

$(8-p)^2 \le 4(16-p^2) \Rightarrow p \le \frac{16}{5}$

4. (d) : $I = \int_0^{\sqrt{e-1}} 1 \cdot dx + \int_{\sqrt{e-1}}^{2} \ln(1+x^2)\, dx$

$= \sqrt{e-1} + x \ln(1+x^2)\Big|_{\sqrt{e-1}}^{2} - 2\int_{\sqrt{e-1}}^{2}\left(1 - \frac{1}{1+x^2}\right)dx$

$= \sqrt{e-1} + 2\ln 5 - \sqrt{e-1} - 4 + 2\sqrt{e-1}$

$+ 2\tan^{-1}2 - 2\tan^{-1}\sqrt{e-1}$

$\Rightarrow$ desired answer = -4

5. (b) : Probability $= \binom{7}{3} 4! \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) \div 7! = \frac{1}{16}$

6. (a, d) : $\frac{\overline{AB}}{AB} \pm \frac{\overline{BC}}{BC} \Rightarrow$ direction ratios are $-1, 2, 1$ and $2, -1, 4$

7. (c): $f - xf' = g - xg' \Rightarrow f - g = cx$

$x + ff' = x + gg' \Rightarrow f^2 - g^2 = d$

$\therefore \; f + g = \frac{d}{cx}, \; f = \frac{1}{2}\left(cx + \frac{d}{cx}\right)$

$g = \frac{1}{2}\left(\frac{d}{cx} - cx\right), \; f(1) = 1, \; g(2) = 3$

$\therefore \; f(x) = \frac{2}{x} - x, \; g(x) = \frac{2}{x} + x$

$f(2) + g(1) = 2$

8. (b) : $\int_1^2 f(x)\, g(x)\, dx = \int_1^2 \left(\frac{4}{x^2} - x^2\right) dx = -\frac{1}{3}$

9. (6): C_1 : centre (0, 0), radius 1

C_2 : centre (3, 0), radius 2

C_3 : centre (0, 4), radius 3

C_4 : centre (α, β), radius r

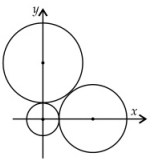
$\alpha^2 + \beta^2 = (r-1)^2$

$(\alpha - 3)^2 + \beta^2 = (r-2)^2$,

$\alpha^2 + (\beta - 4)^2 = (r-3)^2$

On solving, we get

$\alpha = 3, \beta = 4, r = 6$



10. (a):

P. Digits 1 to 9,

$N = \binom{9}{4} = 126, \; S = 9$

Q. Digits 0 to 10,

$N = \binom{10}{4} = 210, \; S = 3$

R. Digits 1 to 9,

$N = \binom{9+4-1}{4} = \binom{12}{4} = 495, \; S = 18$

S. Digits 0 to 10,

$N = \binom{10+4-1}{4} - 1 = \binom{13}{4} - 1 = 714, \; S = 12$

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Solution Sender of Maths Musing

SET-139

1. Khwaja Sami Baig : Azamgarh (U.P.)

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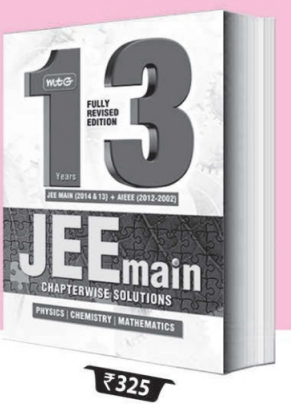

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Q.1 Find the coordinates of the point on the line $y = 4 - 3x$, where xy has a stationary value. Calculate this stationary value and determine its nature. – *Yatin Sharma, Punjab*

Ans. Let $A = xy$

Substitute $y = 4 - 3x$ into $A = xy$

$\therefore \quad A = x(4 - 3x) \quad \Rightarrow \quad A = 4x - 3x^2 \quad \ldots (i)$

Differentiate (i) w.r.t x, we get $\dfrac{dA}{dx} = 4 - 6x$

For stationary points, $\dfrac{dA}{dx} = 0$

$\Rightarrow \quad 4 - 6x = 0 \quad \Rightarrow \quad 6x = 4 \quad \Rightarrow \quad x = \dfrac{2}{3}$

When $x = \dfrac{2}{3}, \quad y = 4 - 3\left(\dfrac{2}{3}\right) = 2$

$\therefore$ Coordinates of stationary point on the given line is $\left(\dfrac{2}{3}, 2\right)$

Now, $A = \dfrac{2}{3}(2) = \dfrac{4}{3}$ and $\dfrac{d^2A}{dx^2} = -6 < 0$

$\therefore$ Stationary value of $xy = \dfrac{4}{3}$ is maximum.

Q.2 Find the number of real roots of the equation

$x^{12} - x^{11} + x^{10} - \ldots - x + 1 = 0 \qquad \ldots(1)$

– *Amrita Bansal, U.P.*

Ans. Multiplying the equation (1) by $x + 1$, the equation becomes $x^{13} + 1 = 0 \Rightarrow x^{13} = -1$

$= \cos(2k + 1)\pi \pm i \sin(2k + 1)\pi$

$\Rightarrow \quad x = \cos\left(\dfrac{2k+1}{13}\right)\pi \pm i \sin\left(\dfrac{2k+1}{13}\right)\pi$

where $k = 0, 1, 2, 3, 4, 5, 6$

For $k = 6$, $x = -1$ is not a root of (1), rest of the values are roots of (1).

Q.3 If $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1 - \cos x \\ 1 + \sin x & \cos x & 1 + \sin x - \cos x \\ \sin x & \sin x & 1 \end{vmatrix}$,

then find the value of $\int_0^{\pi/2} \Delta(x)\,dx$.

– *Madhav Iyer, Thrissur*

Ans. Applying $C_1 \to C_1 - C_2 - C_3$, we get

$$\Delta(x) = \begin{vmatrix} 0 & \cos x & 1 - \cos x \\ 0 & \cos x & 1 + \sin x - \cos x \\ -1 & \sin x & 1 \end{vmatrix}$$

Expanding along C_1, we get

$(-1)\cos x(1 + \sin x - \cos x - 1 + \cos x)$

$= -\cos x \sin x = -\dfrac{1}{2}(\sin 2x)$

$\therefore \quad \int_0^{\pi/2} \Delta(x)dx = \left(-\dfrac{1}{2}\right)\dfrac{(-1)}{2}\left[\cos 2x\right]_0^{\pi/2} = -\dfrac{1}{2}$

Q.4 In a certain test, there are n questions. In this test 2^k students gave wrong answers to atleast $(n - k)$ questions, where $k = 0, 1, 2, \ldots, n$. If the total number of wrong answers is 4095, then find the value of n. – *Adhira Venkatesh, Trivandrum*

Ans. The number of students answering atleast r questions incorrectly is 2^{n-r}.

$\therefore$ The number of students answering exactly $r(1 \le r \le n - 1)$ questions incorrectly is $2^{n-r} - 2^{n-(r+1)}$. Also, the number of students answering all questions wrongly is $2^0 = 1$.

Thus, the total number of wrong answers is

$1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3}) + 3(2^{n-3} - 2^{n-4})$

$+ \ldots + (n - 1)(2^1 - 2^0) + n(2^0)$

$= 2^{n-1} + 2^{n-2} + \ldots + 2^0 = 2^n - 1.$

Now, $2^n - 1 = 4095 \Rightarrow 2^n = 4096 = 2^{12} \Rightarrow n = 12.$

Q.5 Find the sum of 'n' terms of the series.

$$\log_2\left(\frac{x}{y}\right) + \log_4\left(\frac{x}{y}\right)^2 + \log_8\left(\frac{x}{y}\right)^3 + \ldots$$

– *Shubrto Ghosh, W.B.*

Ans. The given equation can be written as

$= \log_2\left(\dfrac{x}{y}\right) + \log_2\left(\dfrac{x}{y}\right) + \log_2\left(\dfrac{x}{y}\right) + \ldots$

$= \log_2\left(\dfrac{x}{y} \times \dfrac{x}{y} \times \dfrac{x}{y} \times \ldots n \text{ times}\right)$

$= \log_2\left(\dfrac{x}{y}\right)^n = n\log_2\left(\dfrac{x}{y}\right)$

OLYMPIAD CORNER

CHALLENGING PROBLEMS FOR RMO, InMO, IMO & OTHER CONTESTS

1. 2500 chess kings have to be placed on a 100×100 chessboard so that

(i) no king can capture any other one (*i.e.*, no two kings are placed in two squares sharing a common vertex);

(ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

2. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA and AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

3. Let $\mathbb{N}$ be the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that the number $(f(m) + n)(m + f(n))$ is a square for all $m, n \in \mathbb{N}$.

4. A sequence $x_1, x_2, \ldots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k$, $x_{2k-1} = (-1)^{k+1} x_k$ for all $k \geq 1$. Prove that $x_1 + x_2 + \ldots + x_n \geq 0$ for $n \geq 1$.

5. $ABCDE$ is a convex pentagon in the plane. Through each vertex draw a straight line which cuts the pentagon into two parts of the same area. Prove that for some vertex, the line through it must intersect the "opposite side" of the pentagon. (Here the opposite side to vertex A is the side CD, the opposite side to B is DE, and so on.)

SOLUTIONS

1. There are two such arrangements.

Suppose that we have an arrangement satisfying the problem conditions. Divide the board into 2×2 pieces; we call these pieces blocks. Each block can contain not more than one king (otherwise these two kings would attack each other); hence, by the pigeonhole principle each block must contain exactly one king.

Now assign to each block a letter T or B if a king is placed in its top or bottom half, respectively. Similarly, assign to each block a letter L or R if a king stands in its left or right half. So we define T-blocks, B-blocks, L-blocks and R-blocks. We also combine the letters; we call a block a TL-block if it is simultaneously T-block and L-block. Similarly we define TR-blocks, BL-blocks and BR-blocks. The arrangement of blocks determines uniquely the arrangement of kings; so in the rest of the solution we consider the 50×50 system of blocks (see Fig. 1). We identify the blocks by their coordinate pairs; the pair (i, j), where $1 \leq i, j \leq 50$, refers to the j^{th} block in the i^{th} row (or the ith block in the j^{th} column). The upper-left block is (1, 1).

The system of blocks has the following properties :

(i) If (i, j) is a B-block then $(i + 1, j)$ is a B-block : otherwise the kings in these two blocks can take each other. Similarly: if (i, j) is a T-block then $(i - 1, j)$ is a T-block; if (i, j) is a L-block then $(i, j - 1)$ is a L-block; if (i, j) is a R-block then $(i, j + 1)$ is a R-block.

(ii) Each column contains exactly 25 L-blocks and 25 R-blocks, and each row contains exactly 25 T-blocks and 25 B-blocks. In particular, the total number of L-blocks (or R-blocks, or T-blocks, or B-blocks) is equal to $25 \cdot 50 = 1250$.

Consider any B-blocks of the form $(1, j)$. By (i), all blocks in the j^{th} column are B-blocks; so we call such a column B-column. By (ii), we have 25 B-blocks in the first row, so we obtain 25 B-columns. These 25 B-columns

contains 1250 B-blocks, hence all blocks in the remaining columns are T-blocks, and we obtain 25 T-columns. Similarly, there are exactly 25 L-rows and exactly 25 R-rows.

Now consider an arbitrary pair of a T-column and a neighbouring B-column (columns with numbers j and $j + 1$).

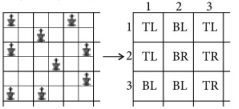

Fig-1

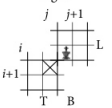

Fig-2

Case 1: Suppose that the j^{th} column is a T-column, and the $(j + 1)^{th}$ column is a B-column. Consider some index i such that the i^{th} row is an L-row; then $(i, j + 1)$ is a BL-block. Therefore, $(i + 1, j)$ cannot be a TR-block (see Fig. 2), hence $(i + 1, j)$ is a TL-block, thus the $(i + 1)^{th}$ row is a L-row. Now, choosing the i^{th} row to be the topmost L-row, we successively obtain that all rows from the i^{th} to the 50^{th} are L-rows. Since we have exactly 25 L-rows, it follows that the rows from the 1^{st} to the 25^{th} are R-rows, and the rows from the 26^{th} to the 50^{th} are L-rows.

Now consider the neighbouring R-row and L-row (that are the rows with numbers 25 and 26). Replacing in the previous reasoning rows by columns and vice versa, the columns from the 1^{st} to the 25^{th} are T-columns, and the columns from the 26^{th} to the 50^{th} are B-columns. So we have a unique arrangement of blocks that leads to the arrangement of kings satisfying the condition of the problem (see Fig. 3).

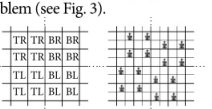

Fig-3

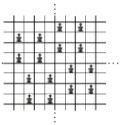

Fig-4

Case 2: Suppose that the j^{th} column is a B-column, and the $(j + 1)^{th}$ column is a T-column. Repeating the arguments from Case 1, we obtain that the rows from the 1^{st} to the 25^{th} are L-rows (and all other rows are R-rows), the columns from the 1^{st} to the 25^{th} are B-columns (and all other columns are T-columns), so we find exactly one more arrangement of kings (see Fig. 4).

2. The line EF intersects the circumcircle at two points. Depending on the choice of P, there are two different cases to consider.

Case 1: The point P lies on the ray EF (see Fig. 1).

Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$.

The quadrilaterals $BCEF$ and $CAFD$ are cyclic due to the right angles at D, E and F. So,

$$\angle BDF = 180° - \angle FDC = \angle CAF = \alpha,$$

$$\angle AFE = 180° - \angle EFB = \angle BCE = \gamma,$$

$$\angle DFB = 180° - \angle AFD = \angle DCA = \gamma.$$

Since P lies on the arc AB of the circumcircle, $\angle PBA < \angle BCA = \gamma$.

Hence, we have

$$\angle PBD + \angle BDF = \angle PBA + \angle ABD + \angle BDF < \gamma + \beta + \alpha = 180°,$$

and the point Q must lie on the extensions of BP and DF beyond the points P and F, respectively.

From the cyclic quadrilateral $APBC$, we get

$$\angle QPA = 180° - \angle APB = \angle BCA = \gamma = \angle DFB = \angle QFA.$$

Hence, the quadrilateral $AQPF$ is cyclic. Then $\angle AQP = 180° - \angle PFA = \angle AFE = \gamma$.

We have obtained that $\angle AQP = \angle QPA = \gamma$, so the triangle AQP is isosceles, $AP = AQ$.

Case 2: The point P lies on the ray FE (see Fig. 2). In this case the point Q lies inside the segment FD.

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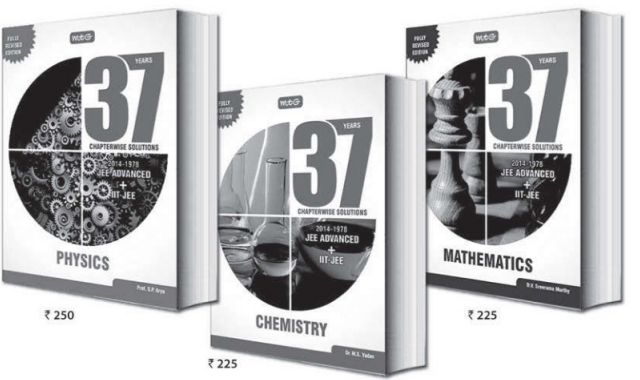

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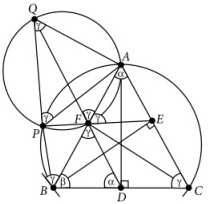


Fig-1

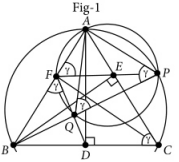


Fig-2

Similarly to the first case, we have $\angle QPA = \angle BCA = \gamma = \angle DFB = 180° - \angle AFQ$. Hence, the quadrilateral $AFQP$ is cyclic.

Then $\angle AQP = \angle AFP = \angle AFE = \gamma = \angle QPA$. The triangle AQP is isosceles again, $\angle AQP = \angle QPA$ and thus $AP = AQ$.

3. First, it is clear that all functions of the form $f(n) = n + c$ with a constant non-negative integer c satisfy the problem conditions since $(f(m) + n)(f(n) + m) = (n + m + c)^2$ is a square.

We are left to prove that there are no other functions. We start with the following Lemma.

Suppose that $p \mid f(k) - f(l)$ for some prime p and positive integers k, l. Then $p \mid k - l$.

Proof : Suppose that $p^2 \mid f(k) - f(l)$, so $f(l) = f(k) + p^2a$ for some integer a. Take some positive integer $D > \max\{f(k), f(l)\}$ which is not divisible by p and set $n = pD - f(k)$. Then the positive numbers $n + f(k) = pD$ and $n + f(l) = pD + (f(l) - (f(k)) = p(D + pa)$ are both divisible by p but not by p^2. Now, applying the problem conditions, we get that both the numbers $(f(k) + n)(f(n) + k)$ and $(f(l) + n)(f(n) + l)$ are squares divisible by p (and thus by p^2); this means that the multipliers $f(n) + k$ and $f(n) + l$ are also divisible by p, therefore $p \mid (f(n) + k) - (f(n) + l) = k - l$ as well.

On the other hand, if $f(k) - f(l)$ is divisible by p but not by p^2, then choose the same number D and set $n = p^3D - f(k)$. Then the positive numbers $f(k) + n = p^3D$ and $f(l) + n = p^3D + (f(l) - f(k))$ are respectively divisible by p^3 (but not by p^4) and by p (but not by p^2). Hence in analogous way we obtain that the numbers $f(n) + k$ and $f(n) + l$ are divisible by p, therefore $p \mid (f(n) + k) - (f(n) + l) = k - l$.

We turn to the problem. First, suppose that $f(k) = f(l)$ for some $k, l \in \mathbb{N}$. Then by Lemma we have that $k - l$ is divisible by every prime number, so $k - l = 0$ or $k = l$. Therefore, the function f is injective.

Next, consider the numbers $f(k)$ and $f(k + 1)$. Since the number $(k + 1) - k = 1$ has no prime divisors, by Lemma the same holds for $f(k + 1) - f(k)$; thus $|f(k + 1) - f(k)| = 1$.

Now, let $f(2) - f(1) = q$, $|q| = 1$. Then we prove by induction that $f(n) = f(1) + q(n - 1)$. The base for $n = 1, 2$ holds by the definition of q. For the step, if $n > 1$ we have $f(n + 1) = f(n) \pm q = f(1) + q(n - 1) \pm q$. Since $f(n) \neq f(n - 2) = f(1) + q(n - 2)$, we get $f(n) = f(1) + qn$, are desired.

Finally, we have $f(n) = f(1) + q(n - 1)$. Then q cannot be -1 since otherwise for $n \geq f(1) + 1$ we have $f(n) \leq 0$ which is impossible. Hence $q = 1$ and $f(n) = (f(1) - 1) + n$ for each $n \in \mathbb{N}$, and $f(1) - 1 \geq 0$, as desired.

4. We start with some observations. First, from the definition of x_i it follows that for each positive integer k, we have

$$x_{4k-3} = x_{2k-1} = -x_{4k-2}$$
and $x_{4k-1} = x_{4k} = -x_{2k} = x_k$. …(1)

Hence, denoting $S_n = \sum_{i=1}^{n} x_i$, we have

$$S_{4k} = \sum_{i=1}^{k} ((x_{4k-3} + x_{4k-2}) + (x_{4k-1} + x_{4k}))$$

$$= \sum_{i=1}^{k} (0 + 2x_k) = 2S_k \quad \ldots(2)$$

$$S_{4k+2} = S_{4k} + (x_{4k+1} + x_{4k+2}) = S_{4k}. \quad \ldots(3)$$

Observe also that $S_n = \sum_{i=1}^{n} x_i \equiv \sum_{i=1}^{n} 1 = n \pmod 2$.

Now we prove by induction on k that $S_i \geq 0$ for all $i \leq 4k$. The base case is valid since $x_1 = x_3 =$

$x_4 = 1, x_2 = -1$. For the induction step, assume that $S_i \geq 0$ for all $i \leq 4k$. Using the relations (1) – (3), we obtain

$S_{4k+4} = 2S_{k+1} \geq 0$, $S_{4k+2} = S_{4k} \geq 0$,

$S_{4k+3} = S_{4k+2} + x_{4k+3} = \dfrac{S_{4k+2} + S_{4k+4}}{2} \geq 0.$

So, we are left to prove that $S_{4k+1} \geq 0$. If k is odd, then $S_{4k} = 2S_k \geq 0$; since k is odd, S_k is odd as well, so we have $S_{4k} \geq 2$ and hence $S_{4k+1} = S_{4k} + x_{4k+1} \geq 1$.

Conversely, if k is even, then we have $x_{4k+1} = x_{2k+1} = x_{k+1}$, hence $S_{4k+1} = S_{4k} + x_{4k+1} = 2S_k + x_{k+1} = S_k + S_{k+1} \geq 0$. The step is proved.

5. We denote the area of the polygon $A_1 A_2 \ldots A_n$ by $[A_1 A_2 \ldots A_n]$, and we put $[ABCDE] = S$. We assume that the straight line through A which bisects the area of the pentagon does not intersect the opposite side CD.

Then either $[ABC] = \dfrac{1}{2}S$ or $[ADE] > \dfrac{1}{2}S$.

We may assume without loss of generality that

$[ABC] > \dfrac{1}{2}S.$ …(1)

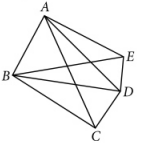
Now we assume that the straight line through B which bisects the area of the pentagon does not intersect the opposite side DE. Then either

$[BAE] > \dfrac{1}{2}S$ or $[BCD] > \dfrac{1}{2}S$.

Case 1.: $[BAE] > \dfrac{1}{2}S.$ …(2)

From (1) we get that $[ADE] < [ACDE] < \dfrac{1}{2}S$

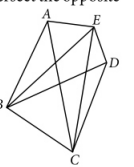
and from (2) we get $[BCD] < [BCDE] < \dfrac{1}{2}S$.



Therefore the line through D which bisects the area of the pentagon must intersect the opposite side AB.

Case 2.: $[BCD] > \dfrac{1}{2}S.$ …(3)

From (1) we get $[ECD] < [ACDE] < \dfrac{1}{2}S$ and $[EAB] < [EABD] < \dfrac{1}{2}S$.



Therefore the line through E which bisects the area of the pentagon must intersect the opposite side BC.

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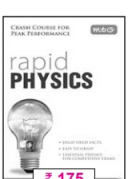

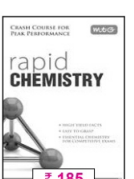

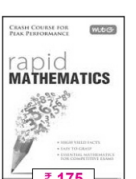

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Matrices and Determinants

MATRICES

A rectangular arrangement of mn elements in the form of an ordered set of m rows, each row consisting of an ordered set of n elements is called an $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

or, $A = [a_{ij}]_{m\times n}$, $1 \le i \le m$, $1 \le j \le n$

So, A is called a matrix of order (or type) $m\times n$.

COMPARABLE MATRICES

Two matrices are said to be comparable iff any one of them contains as many rows and columns as the other.

EQUAL MATRICES

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal (written as $A = B$) iff

(i) they are of the same order, *i.e.* number of rows in A = number of rows in B and number of columns in A = number of columns in B, and

(ii) their corresponding elements are equal *i.e.*, $a_{ij} = b_{ij}$ for all i and j.

TYPES OF MATRICES

	Types	Definitions
1.	Rectangular matrix	If the number of rows and columns of a matrix are not equal, then the matrix is called a rectangular matrix, *i.e.*, $m \neq n$.
2.	Row matrix	A matrix containing only one row is called a row matrix *i.e.*, $m = 1$
3.	Column matrix	A matrix containing only one column is called a column matrix *i.e.*, $n = 1$
4.	Null or zero matrix	If every element of a matrix is zero, then it is called a zero or null matrix.
5.	Square matrix	A matrix in which the number of rows is equal to the number of columns is called a square matrix. *i.e.*, $m = n$
6.	Diagonal matrix	A square matrix in which every non diagonal element is zero is called a diagonal matrix.
7.	Scalar matrix	A square matrix in which all the elements in the diagonal are equal and rest of the elements are zero is called a scalar matrix.
8.	Identity or unit matrix	A square matrix in which each diagonal element is unity and all other elements are zero is called a unit matrix. It is denoted by I.
9.	Upper triangular matrix	A square matrix in which all the elements below the diagonal are zero is called an upper triangular matrix.

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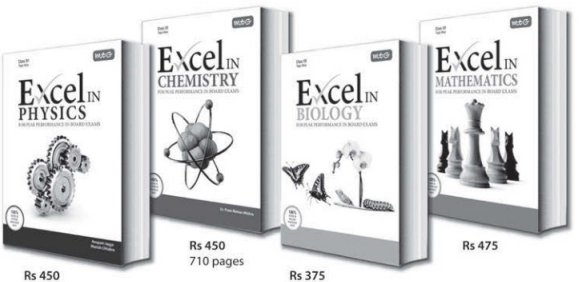

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10.	Lower triangular matrix	A square matrix in which all the elements above the diagonal are zero is called a lower triangular matrix.

OPERATION ON MATRICES

- **Matrix addition :** Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the sum $A + B$ of A and B is defined as $A + B = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$ for each i and j.
- **Subtraction of a matrix :** Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ be two matrices of same order, then the difference $A - B$ of A and B is defined as $A - B = A + (-B) = [a_{ij} - b_{ij}]_{m \times n}$ for each i and j.
- **Negative of a matrix :** Let $A = [a_{ij}]_{m \times n}$ be a matrix, then the negative of the matrix A is defined as the matrix $[-a_{ij}]_{m \times n}$ and is denoted by $-A$.
- **Scalar multiplication of a matrix :** Let $[a_{ij}]_{m \times n}$ be a matrix and k is a scalar, then the matrix obtained by multiplying the individual elements of the matrix A by k is called the scalar multiplication of A by k and is denoted by kA or Ak.
- **Multiplication of two matrices :** Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{jk}]_{n \times p}$ be two matrices, then the product $AB = C = [c_{ik}]_{m \times p}$ is called multiplication of two matrices where $c_{ik} = a_{i1}\, b_{1k} + a_{i2}\, b_{2k} + ... + a_{in}\, b_{nk}$.

Properties of Matrix Addition and Multiplication

- **Commutative law**
  If A and B are two matrices, then
  $A + B = B + A$ and $AB \neq BA$
- **Associative law**
  If A, B and C are three matrices, then
  $(A + B) + C = A + (B + C)$ and $(AB)C = A(BC)$
- **Distributive law**
  $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$
- **Existence of identity**
  The null matrix is the identity element for matrix addition *i.e.*, $A + O = A = O + A$
  If A is a square matrix, then identity element for matrix multiplication is I, *i.e.*, $IA = A = AI$, of same order.
- **Existence of inverse**
  For every matrix $A = [a_{ij}]_{m \times n}$, there exists a matrix $[-a_{ij}]_{m \times n}$ denoted by $-A$, such that $A + (-A) = O = (-A) + A$
- If A is a $m \times n$ matrix and O is a null matrix, then
  (i) $A_{m \times n} \times O_{n \times p} = O_{m \times p}$
  (ii) $O_{p \times m} \times A_{m \times n} = O_{p \times n}$

POWERS OF A SQUARE MATRIX

Let A be a square matrix of order n, then AA makes sense and it is also a square matrix of order n. We define :

$A^1 = A$
$A^2 = AA$
… …
… …
$A^m = A^{m-1}\, A = AA^{m-1}$, for all positive integers m.

MATRIX POLYNOMIAL

Let $f(x) = a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m$ be a polynomial in x and A be a square matrix of order n, then

$f(A) = a_0A^m + a_1A^{m-1} + \ldots + a_{m-1}A + a_mI_n$ is called a matrix polynomial in A. Thus, to obtain $f(A)$, replace x by A in $f(x)$ and the constant term is multiplied by the identity matrix of the order equal to that of A.

The polynomial equation $f(x) = 0$ is said to be satisfied by the matrix A iff $f(A) = O$.

TRANSPOSE OF A MATRIX

- Let $A = [a_{ij}]_{m \times n}$ be a matrix. Then the transpose of A, denoted by A^T or A', is a $n \times m$ matrix such that $A^T = [a_{ji}]_{n \times m}$ $\forall$ $i = 1, 2,....., m$ and $j = 1, 2,...., n$.

  Thus, A^T is obtained from A by changing its rows into columns and its columns into rows.

Properties of Transpose

- Let A and B be two matrices. Then
  (i) $(A^T)^T = A$
  (ii) $(A + B)^T = A^T + B^T$, where A and B are of the same order.
  (iii) $(kA)^T = kA^T$, where k be any scalar (real or complex)
  (iv) $(AB)^T = B^TA^T$, where A and B being conformable for the product AB (Reversal law).
  (v) $(ABC)^T = C^T B^T A^T$.

ELEMENTARY OPERATIONS

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns, which are known as elementary operations or transformations.

(i) The multiplication of the elements of any row by a non zero number k is denoted by $R_i \to kR_i$
The corresponding column operation is denoted by $C_i \to kC_i$ where $k \neq 0$

For example, applying $R_2 \to \frac{1}{4} R_2$ to

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ we get } \begin{bmatrix} a_{11} & a_{12} \\ \frac{a_{21}}{4} & \frac{a_{22}}{4} \end{bmatrix}$$

(ii) The interchange of any two rows or two columns is denoted by, $R_i \leftrightarrow R_j$ and $C_i \leftrightarrow C_j$.

For example, applying $R_1 \leftrightarrow R_3$ to

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \text{we get } \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

(iii) The addition to the elements of any row or column, to the corresponding elements of any other row or column multiplied by any non zero number k is denoted by
$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$.
For example, applying $R_1 \to R_1 - kR_2$ to

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ we get} \begin{bmatrix} a_{11} - ka_{21} & a_{12} - ka_{22} \\ a_{21} & a_{22} \end{bmatrix}$$

SOME SPECIAL MATRICES

- **Symmetric matrix :** A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$ for all i, j.
- **Skew-symmetric matrix :** A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$ for all i, j.
- **Orthogonal matrix :** A square matrix A is called an orthogonal matrix if $AA^T = A^T A = I$.

INVERTIBLE MATRICES

If A is a square matrix of order n and if there exists another square matrix B of the same order n, such that $AB = BA = I$, then B is called the inverse matrix of A and it is denoted by A^{-1}. In that case, A is said to be invertible.

Uniqueness of inverse

Inverse of a square matrix, if it exists, is unique.

Let $A = [a_{ij}]$ be a square matrix of order n. If possible, let B and C be two inverses of A.
To show : $B = C$.

Since B is the inverse of A

$\Rightarrow \quad AB = BA = I \quad \ldots (1)$

Since C is also the inverse of A

$\Rightarrow \quad AC = CA = I \quad \ldots (2)$

Thus, $B = BI = B\,(AC) = (BA)\,C = IC = C$

If A and B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1}\,A^{-1}$.

Inverse of a Matrix by Elementary Operations

To find A^{-1} using elementary row operations, write $A = IA$ and apply a sequence of row operations on $A = IA$ till we get, $I = BA$. The matrix B will be the inverse of A. Similarly, if we wish to find A^{-1} using column operations, then write $A = AI$ and apply a sequence of column operations on $A = AI$ till we get, $I = AB$.

DETERMINANTS

To every square matrix $A = [a_{ij}]$ of order n, $a_{ij} \in R$, we can associate a unique real number called determinant of matrix A, denoted by det A or $|A|$. This may be thought as a function which associate each square matrix with a unique real number *i.e.*, $f : M \to R$ given by $f(A) = \det A = |A|$, where M is the set of square matrices and R is the set of real numbers.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then determinant of A is written as

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Sometimes, a determinant is denoted by Δ. Note that only square matrices have determinants.

- **Value of determinant of order 1**
  Let $A = [a]$ be a matrix of order 1, then determinant of A is equal to a.
- **Value of determinant of order 2**
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{21}a_{12})$$
- **Value of determinant of order 3 or more :**
  The value of a determinant is the sum of the product of a row (or a column) with their corresponding cofactors.
$$\therefore \quad \det A = \sum_{i=1}^{3} a_{ij}\, C_{ij}$$
- **Value of determinant of order n :**
$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij}$$
- **Properties of Determinants**
  - **(i)** $\det I_n = 1$, where I_n is unit/identity matrix of order n.
  - **(ii)** $\det O_n = 0$, O is null matrix (square matrix of n order).
  - **(iii)** $A = [a_{ij}]_{n \times n}$ then $|A| = |A'|$ {Reflection Property}.
  - **(iv)** $\det (AB) = \det A \cdot \det B$, where A & B are matrices of same order.
  - **(v)** $\det (kA) = k^n \det A$, if A is of order $n \times n$.
  - **(vi)** $\det (A^n) = (\det A)^n$ if $n \in I^+$.
  - **(vii)** $|A| = 0$ iff
    - (a) Any two rows or columns are identical.
    - (b) Any two rows or columns are in proportion.
    - (c) Each element of any row/column is zero.
  - **(viii)** If each element of a row/column of a determinant is multiplied by k then value of new determinant is k times the original determinant.
  - **(ix)** Determinant of a diagonal matrix is the product of its diagonal elements.
  - **(x)** If two rows/columns of a determinant are interchanged, then the determinant retains its absolute value but its sign is changed.
  - **(xi)** If each element of a row/column of a determinant is expressed as a sum of two or more terms, then determinant can be expressed as the sum of two or more determinants.
  - **(xii)** If in any row/column of a determinant, a multiple of another row/column is added, then the value of determinant does not change.
  - **(xiii)** The sum of product of the elements of any row/column of a determinant with cofactors of the corresponding elements of any other row/column is zero.

APPLICATION OF DETERMINANTS

- Area of triangle with vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.
- If area is zero then three points are collinear.

MINORS AND COFACTORS

- **Minor of a_{ij} in A :** The minor of an element a_{ij} in A is defined as the value of the determinant obtained by deleting the i^{th} row and j^{th} column of A, and it is denoted by M_{ij}.
- **Cofactor of a_{ij} in A :** The cofactor C_{ij} of an element a_{ij} is defined as $C_{ij} = (-1)^{i+j} M_{ij}$.

ADJOINT OF A MATRIX

- Adjoint of a matrix $A = [a_{ij}]_{n \times n}$ is defined as $\text{adj } A = [C_{ij}]_{n \times n}$, where C_{ij} represents cofactor of a_{ij} in A.

Properties of Adjoint of a Matrix

- For every square matrix $A(\text{adj } A) = (\text{adj } A)A = |A| \cdot I$
- If $|A| = 0$, matrix A is called singular else non singular.
- If A and B are non singular matrices of the same order, then AB and BA are also non singular matrices of same order.
- $(\text{adj } AB) = (\text{adj } B) \cdot (\text{adj } A)$
- $(\text{adj } A)' = \text{adj } A'$
- Let A be $n \times n$ matrix, then
  (i) $|\text{adj } A| = |A|^{n-1}$
  (ii) $\text{adj}(\text{adj } A) = |A|^{n-2} A$
  (iii) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

INVERSE OF A MATRIX

A non-zero matrix A of order n is said to be invertible if there exists a square matrix B of order n such that $AB = BA = I$. We say $A^{-1} = B$.

- A matrix A is invertible if $|A| \neq 0$.
- $A^{-1} = \frac{1}{|A|} \cdot (\text{adj } A)$.
- If $AB = AC \Rightarrow B = C$ if $|A| \neq 0$.
- $(AB)^{-1} = B^{-1}A^{-1}$.
- $(A^T)^{-1} = (A^{-1})^T$.
- If A is an invertible symmetric matrix then A^{-1} is also symmetric.

SOLUTION OF SYSTEM OF EQUATIONS

- Let $AX = B$ be the given system of equations :
  - **(i)** If $|A| \neq 0$, the system is consistent and has one unique solution.
  - **(ii)** If $|A| = 0$ and (adj $A)B \neq O$, then the system is inconsistent so it has no solution.
  - **(iii)** If $|A| = 0$ and (adj $A)B = O$, then the system is consistent but has infinitely many solutions.

VERY SHORT ANSWER TYPE (1 MARK)

1. Evaluate : $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$

2. If the matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{bmatrix}$ is singular, find x.

3. Construct a 2×3 matrix whose elements a_{ij} are given by $a_{ij} = 2i - 3j$

4. If $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$, then find the value of k.

SHORT ANSWER TYPE (4 MARKS)

5. If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then find k so that $A^2 = 8A + kI$

6. Without expanding, show that $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0.$

7. Find the equation of the line joining $A(1, 3)$ and $B(0, 0)$ using determinants and find k if $D(k, 0)$ is a point such that area of ΔABD is 3 sq. units.

8. Use matrix multiplication to divide ₹ 30,000 in two parts such that the total annual interest at 9% on the first part and 11% on the second part amounts ₹ 3060.

9. Show that $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$

10. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix which satisfies $AA^t = 9I_3$, then find the values of a and b.

LONG ANSWER TYPE (6 MARKS)

11. Express the following matrix as the sum of a symmetric matrix and a skew-symmetric matrix and verify your result : $\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

12. Using properties of determinants, prove the following :

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1 + pxyz)(x - y)(y - z)(z - x).$$

13. Solve the following system of equations, using matrices :

$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$, $\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$, $\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$; $x, y, z \neq 0$.

14. Using elementary row transformation, find the inverse of the matrix $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$

15. An amount of ₹ 5000, is put into three investments at the rate of interest of 6%, 7% and 8% per annum respectively. The total annual income is ₹ 358. If the combined income from the first two investments is ₹ 70 more than the income from the third, find the amount of each investment by matrix method.

SOLUTIONS

1. $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$

$= \cos 15^\circ \times \cos 75^\circ - \sin 15^\circ \times \sin 75^\circ$

$= \cos(15^\circ + 75^\circ)$

$(\because \cos x \cos y - \sin x \sin y = \cos(x + y))$

$= \cos 90^\circ = 0$

2. Given that matrix A is singular $\Rightarrow |A| = 0$

$\Rightarrow \begin{vmatrix} 1 & -2 & 3 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$ (Operate $R_1 \to R_1 + R_3$)

$\Rightarrow \begin{vmatrix} x+1 & 0 & 0 \\ 1 & 2 & 1 \\ x & 2 & -3 \end{vmatrix} = 0$ (Expand by R_1)

$\Rightarrow (x + 1)(-6 - 2) = 0$

$\Rightarrow -8(x + 1) = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1$

3. $a_{ij} = 2i - 3j$, $i = 1, 2$ and $j = 1, 2, 3$

$\therefore \quad a_{11} = 2\cdot1 - 3\cdot1 = -1,$

$a_{12} = 2\cdot1 - 3\cdot2 = -4,$

$a_{13} = 2\cdot1 - 3\cdot3 = -7,$

$a_{21} = 2\cdot2 - 3\cdot1 = 1,$

$a_{22} = 2\cdot2 - 3\cdot2 = -2,$

$a_{23} = 2\cdot2 - 3\cdot3 = -5.$

Hence, the required matrix is $\begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \end{bmatrix}$.

4. Given $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1\cdot3+2\cdot2 & 1\cdot1+2\cdot5 \\ 3\cdot3+4\cdot2 & 3\cdot1+4\cdot5 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 7 & 11 \\ 17 & 23 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ k & 23 \end{bmatrix} \Rightarrow k = 17.$

5. We have, $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A^2 = AA = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix}$

and, $8A + kI = 8\begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix} + k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$

$\therefore A^2 = 8A + kI$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ -8 & 49 \end{bmatrix} = \begin{bmatrix} 8+k & 0 \\ -8 & 56+k \end{bmatrix}$

$\Rightarrow 1 = 8 + k$ and $56 + k = 49 \quad \therefore \quad k = -7.$

6. Let $\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$.

Applying $R_1 \to R_1(a)$, $R_2 \to R_2(b)$ and $R_3 \to R_3(c)$,

we get $\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c^2 & abc & ab+ac \\ bc^2a^2 & abc & bc+ba \\ ca^2b^2 & abc & ac+bc \end{vmatrix}$

$\Rightarrow \Delta = \frac{1}{abc}(abc)^2\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ba \\ ab & 1 & ac+bc \end{vmatrix}$

[Take out abc common from C_1 and C_2]

$\Rightarrow \Delta = abc\begin{vmatrix} bc & 1 & ab+bc+ca \\ ca & 1 & ab+bc+ca \\ ab & 1 & ab+bc+ca \end{vmatrix}$

[Applying $C_3 \to C_3 + C_1$]

$\Rightarrow \Delta = abc\,(ab+bc+ca)\begin{vmatrix} bc & 1 & 1 \\ ca & 1 & 1 \\ ab & 1 & 1 \end{vmatrix}$

[Taking out $ab + bc + ca$ common from C_3]

$\Rightarrow \Delta = abc\,(ab + bc + ca) \times 0 = 0$

[$\because C_2$ and C_3 are identical]

7. Let $P(x, y)$ be a point on line AB. Then,

Area of $\Delta ABP = 0 \Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ x & y & 1 \end{vmatrix} = 0$

$\Rightarrow \frac{1}{2}\{1(0-y)-3(0-x)+1(0-0)\} = 0$

$\Rightarrow 3x - y = 0$. This is the required equation of AB.

Now, Area of $\Delta ABD = 3$ sq. units

$$\Rightarrow \frac{1}{2}\begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 3$$

$$\Rightarrow \begin{vmatrix} 1 & 3 & 1 \\ 0 & 0 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 6$$

$\Rightarrow 1(0-0) - 3(0-k) + 1(0-0) = \pm 6$

$\Rightarrow 3k = \pm 6 \Rightarrow k = \pm 2$

8. Let the two parts be ₹ x and ₹ $(30000 - x)$ respectively. Let A be the 1×2 matrix representing these two parts

Part I Part II

i.e. $A = [x \quad 30000 - x]$

Let R denotes the 2×1 matrix representing the annual interest rates on two parts *i.e.*,

$$R = \begin{matrix} \text{Part I} \\ \text{Part II} \end{matrix}\begin{bmatrix} 0.09 \\ 0.11 \end{bmatrix}$$

The total annual interest on the two parts is given by the matrix multiplication AR.

$\Rightarrow AR = 3060$

$$\Rightarrow [x \quad 30000 - x]\begin{bmatrix} 0.09 \\ 0.11 \end{bmatrix} = 3060$$

$$\Rightarrow 0.09x + 0.11(30000 - x) = 3060$$

$$\Rightarrow \frac{9}{100}x + \frac{11}{100}(30000 - x) = 3060$$

$\Rightarrow 9x + 330000 - 11x = 306000 \Rightarrow x = 12{,}000$

Hence, two parts of ₹ 30,000 are ₹ 12,000 and ₹ 18,000 respectively.

9. Let $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$

Applying $C_2 \to C_2 - pC_1$ and $C_3 \to C_3 - qC_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 1+3p \\ 3 & 6 & 1+6p \end{vmatrix}$$

Applying $C_3 \to C_3 - pC_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 6 & 1 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 1 \end{vmatrix}$$

Expanding along R_1, we get

$\Rightarrow \Delta = 5 - 4 = 1$

10. $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$

Given, $AA^t = 9I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$\Rightarrow a + 4 + 2b = 0, 2a + 2 - 2b = 0, a^2 + 4 + b^2 = 9$

$\Rightarrow a + 2b + 4 = 0 \quad \ldots(i)$

$a - b + 1 = 0 \quad \ldots(ii)$

$a^2 + b^2 = 5 \quad \ldots(iii)$

Solving (i) and (ii) simultaneously for a and b, we get $a = -2$, $b = -1$

11. Let $A = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$

$$\Rightarrow A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

$$\therefore A + A' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix}$$

$$\Rightarrow \quad \frac{1}{2}(A+A')=\begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}$$

which is clearly a symmetric matrix and

$$A-A'=\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}-\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & -5 & -3 \\ 5 & 0 & -6 \\ 3 & 6 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \frac{1}{2}(A-A')=\begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}$$

which is clearly a skew symmetric matrix.

Since $A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$

$$\therefore \quad A=\begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}+\begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}$$

Thus A has been expressed as the sum of a symmetric matrix and a skew-symmetric matrix:

Verification

$$\begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}+\begin{bmatrix} 0 & -\frac{5}{2} & -\frac{3}{2} \\ \frac{5}{2} & 0 & -3 \\ \frac{3}{2} & 3 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}=A.$$

12. By using property, we get

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}=\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+\begin{vmatrix} x & x^2 & px^3 \\ y & y^2 & py^3 \\ z & z^2 & pz^3 \end{vmatrix}$$

(Take p out from C_3 of second determinant)

$$=\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+p\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

(Interchange C_3 with C_1 and C_2 taking x, y and z out from R_1, R_2 and R_3 respectively of second determinant)

$$=\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}+pxyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$=(1+pxyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$=(1+pxyz)\begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

(Take $(x-y)$ out from R_1 and $(y-z)$ out from R_2)

$$=(1+pxyz)(x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$$

(Expand by C_1)

$$=(1+pxyz)(x-y)(y-z)\cdot 1\cdot((y+z)-(x+y))$$

$$=(1+pxyz)(x-y)(y-z)(z-x).$$

13. Let $\frac{1}{x}=a$, $\frac{1}{y}=b$ and $\frac{1}{z}=c$, then the given system of equations become

$$\left.\begin{aligned} 2a+3b+10c&=4 \\ 4a-6b+5c&=1 \\ 6a+9b-20c&=2 \end{aligned}\right\} \qquad \ldots(i)$$

This system of equations can be written as $AX = B$

where, $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$,

$X = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$.

Now $|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2 \cdot 3 \cdot 5 \begin{vmatrix} 1 & 1 & 2 \\ 2 & -2 & 1 \\ 3 & 3 & -4 \end{vmatrix}$

Applying $C_1 \to C_1 - C_2$, we get

$= 30 \begin{vmatrix} 0 & 1 & 2 \\ 4 & -2 & 1 \\ 0 & 3 & -4 \end{vmatrix}$ (Expand by C_1)

$= 30[-4(-4-6)] = 30 \times 40 = 1200 \neq 0.$

$\Rightarrow$ A^{-1} exists and so the system (i) has a unique solution $X = A^{-1}B$.

Here, $A_{11} = \begin{vmatrix} -6 & 5 \\ 9 & -20 \end{vmatrix} = 75$, $A_{12} = -\begin{vmatrix} 4 & 5 \\ 6 & -20 \end{vmatrix} = 110$,

$A_{13} = \begin{vmatrix} 4 & -6 \\ 6 & 9 \end{vmatrix} = 72$;

$A_{21} = -\begin{vmatrix} 3 & 10 \\ 9 & -20 \end{vmatrix} = 150$, $A_{22} = \begin{vmatrix} 2 & 10 \\ 6 & -20 \end{vmatrix} = -100$,

$A_{23} = -\begin{vmatrix} 2 & 3 \\ 6 & 9 \end{vmatrix} = 0$;

$A_{31} = \begin{vmatrix} 3 & 10 \\ -6 & 5 \end{vmatrix} = 75$, $A_{32} = -\begin{vmatrix} 2 & 10 \\ 4 & 5 \end{vmatrix} = 30$,

$A_{33} = \begin{vmatrix} 2 & 3 \\ 4 & -6 \end{vmatrix} = -24.$

$\therefore$ $\text{adj } A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^t = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

$\therefore$ $A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

$\therefore$ $X = A^{-1}B = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$= \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix}$

$\Rightarrow$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix} \Rightarrow a = \frac{1}{2}, b = \frac{1}{3}, c = \frac{1}{5}$

$\Rightarrow$ $\frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3}, \frac{1}{z} = \frac{1}{5} \Rightarrow x = 2, y = 3, z = 5.$

Hence, the solution of the given system of equations is $x = 2$, $y = 3$, $z = 5$.

14. We have $A = IA$

$\Rightarrow$ $\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Applying $R_1 \to R_1 - R_2$, we get

$\Rightarrow$ $\begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$

Applying $R_2 \to R_2 + (-2)R_1$ and $R_3 \to R_3 + (-3)R_1$, we get

$\Rightarrow$ $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{bmatrix} A$

Applying $R_2 \to (1/2)R_2$, we get

$\Rightarrow$ $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3/2 & 0 \\ -3 & 3 & 1 \end{bmatrix} A$

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 + 2R_2$, we get

$\Rightarrow$ $\begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5 & 6 & 1 \end{bmatrix} A$

Applying $R_3 \to 1/4\ R_3$, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 \\ -1 & 3/2 & 0 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A$$

Applying $R_1 \to R_1 + (1/2)R_3$ and

$R_2 \to R_2 - (1/2)\ R_3$, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix} A$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} -5/8 & 5/4 & 1/8 \\ -3/8 & 3/4 & -1/8 \\ -5/4 & 3/2 & 1/4 \end{bmatrix}$$

15. Let x, y and z be the investments at the rates of interest of 6%, 7% and 8% per annum respectively. Then, $x + y + z = 5000$.

Now, Income from first investment of

$$₹\ x = ₹\frac{6x}{100}$$

Income from second investment of

$$₹\ y = ₹\frac{7y}{100}$$

Income from third investment of $₹\ z = ₹\frac{8z}{100}$

$$\therefore \text{ Total annual income} = ₹\left(\frac{6x}{100} + \frac{7y}{100} + \frac{8z}{100}\right)$$

$$\Rightarrow \frac{6x}{100} + \frac{7y}{100} + \frac{8z}{100} = 358$$

$$\Rightarrow 6x + 7y + 8z = 35800.$$

Also, by given condition $\frac{6x}{100} + \frac{7y}{100} = 70 + \frac{8z}{100}$

$$\Rightarrow 6x + 7y - 8z = 7000$$

So, we obtain the following system of linear equations :

$$x + y + z = 5000$$
$$6x + 7y + 8z = 35800$$
$$6x + 7y - 8z = 7000$$

The system of equations can be written in matrix form as:

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & 7 & 8 \\ 6 & 7 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5000 \\ 35800 \\ 7000 \end{bmatrix} \text{ or, } AX = B$$

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 6 & 7 & 8 \\ 6 & 7 & -8 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5000 \\ 35800 \\ 7000 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 7 & 8 \\ 6 & 7 & -8 \end{vmatrix} = -16 \neq 0.$

So, A^{-1} exist and the solution of the given system of equations is given by $X = A^{-1}\ B$.

Let C_{ij} be the cofactor of a_{ij} in $A = [a_{ij}]$.

Then, $C_{11} = -112$, $C_{12} = 96$, $C_{13} = 0$, $C_{21} = 15$, $C_{22} = -14$, $C_{23} = -1$, $C_{31} = 1$, $C_{32} = -2$ and $C_{33} = 1$.

$$\therefore \text{ adj } A = \begin{bmatrix} -112 & 96 & 0 \\ 15 & -14 & -1 \\ 1 & -2 & 1 \end{bmatrix}^T = \begin{bmatrix} -112 & 15 & 1 \\ 96 & -14 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\text{So, } A^{-1} = \frac{1}{|A|}(\text{adj } A) = -\frac{1}{16}\begin{bmatrix} -112 & 15 & 1 \\ 96 & -14 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

Hence, the solution is given by

$$X = A^{-1}\ B = \frac{-1}{16}\begin{bmatrix} -112 & 15 & 1 \\ 96 & -14 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 5000 \\ 35800 \\ 7000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{16}\begin{bmatrix} -560000 + 537000 + 7000 \\ 480000 - 501200 - 14000 \\ 0 - 35800 + 7000 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1000 \\ 2200 \\ 1800 \end{bmatrix}$$

$$\Rightarrow \quad x = 1000,\ y = 2200,\ z = 1800$$

Hence, three investments are of ₹ 1000, ₹ 2200 and ₹ 1800 respectively.

■■

Indian man named joint winner of US prize

Nikhil Srivastava, a young Indian mathematician, has become the first joint winner of the prestigious 2014 George Polya Prize for finding a solution to a problem that had eluded mathematicians for more than half a century. Srivastava, who has a PhD from Yale University and is presently based in Bangalore, is part of a three-member team which found a solution to the Kadison-Singer conjecture that pertains to the mathematical foundations of quantum mechanics.

concept BOOSTERS

Class XI

Quadratic Equations

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

The general form of a quadratic equation in x is $ax^2 + bx + c = 0$, where $a, b, c \in R$ & $a \neq 0$.

Results:

- The solution of the quadratic equation, $ax^2 + bx + c = 0$ is given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The expression $b^2 - 4ac = D$ is called the discriminant of the quadratic equation.
- If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

  (i) $\alpha + \beta = -b/a$ (ii) $\alpha\beta = c/a$

  (iii) $\alpha - \beta = \dfrac{\sqrt{D}}{a}$

NATURE OF ROOTS

(A) Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a, b, c \in R$ and $a \neq 0$ then

- $D > 0 \Leftrightarrow$ roots are real and distinct (unequal).
- $D = 0 \Leftrightarrow$ roots are real and coincident (equal).
- $D < 0 \Leftrightarrow$ roots are imaginary.
- If $p + iq$ is one root of a quadratic equation, then the other must be the conjugate $p - i\,q$ and vice versa. ($p, q \in R$ and $i = \sqrt{-1}$)

(B) Consider the quadratic equation

$$ax^2 + bx + c = 0,$$

where $a, b, c \in Q$ and $a \neq 0$ then;

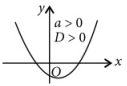
- If $D > 0$ and is a perfect square, then roots are rational and unequal.
- If $\alpha = p + \sqrt{q}$ is one root, in this case, (where p is rational and $\sqrt{q}$ is a surd) then the other root must be the conjugate of it *i.e.* $\beta = p - \sqrt{q}$ and vice versa.

- A quadratic equation whose roots are α and β, is $(x - \alpha)(x - \beta) = 0$ *i.e.* $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ *i.e.* $x^2 -$ (sum of roots)$x +$ product of roots $= 0$.
- Remember that a quadratic equation cannot have three different roots and if it has, it becomes an identity.
- **Consider the quadratic expression,** $y = ax^2 + bx + c$, $a \neq 0$ and $a, b, c \in R$ then
  - The graph between x, y is always a parabola. If $a > 0$, then the shape of the parabola is concave upwards and if $a < 0$ then the shape of the parabola is concave downwards.
  - $\forall\, x \in R$, $y > 0$ only if $a > 0$ and $b^2 - 4ac < 0$.
  - $\forall\, x \in R$, $y < 0$ only if $a < 0$ and $b^2 - 4ac < 0$.

Carefully go through the 6 different shapes of the parabola given below.

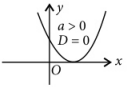
* Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

Fig. 1



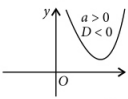
Roots are real and distinct

Fig. 2



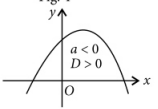
Roots are coincident

Fig. 3



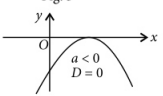
Roots are complex conjugate

Fig. 4



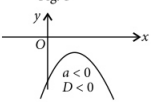
Roots are real and distinct

Fig. 5



Roots are coincident

Fig. 6



Roots are complex conjugate

SOLUTION OF QUADRATIC INEQUALITIES

We have, $ax^2 + bx + c > 0$ $(a \neq 0)$.

- If $D > 0$, then the equation $ax^2 + bx + c = 0$ has two different roots $x_1 < x_2$.
  Then $a > 0 \Rightarrow x \in (-\infty, x_1) \cup (x_2, \infty)$
  $a < 0 \Rightarrow x \in (x_1, x_2)$
- If $D = 0$, then roots are equal, *i.e.* $x_1 = x_2$.
  In that case $a > 0 \Rightarrow x \in (-\infty, x_1) \cup (x_1, \infty)$
  $a < 0 \Rightarrow x \in \phi$
- Inequalities of the form $\frac{P(x)}{Q(x)} \lessgtr 0$ can be quickly solved using the method of intervals.
- **Maximum & minimum value** of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as ;
  $a < 0$ or $a > 0$. $y \in \left[\frac{4ac - b^2}{4a}, \infty\right)$ if $a > 0$ and
  $y \in \left(-\infty, \frac{4ac - b^2}{4a}\right]$ if $a < 0$.

COMMON ROOTS OF TWO QUADRATIC EQUATIONS [ONLY ONE COMMON ROOT] :

- Let α be the common root of $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$. Therefore
  $a\alpha^2 + b\alpha + c = 0$ and $a'\alpha^2 + b'\alpha + c' = 0$.
  By Cramer's Rule
  $$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$
  Therefore, $\alpha = \frac{ca' - c'a}{ab' - a'b} = \frac{bc' - b'c}{a'c - ac'}$
  So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.
- **The condition that a quadratic function** $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors is that
  $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
  or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

THEORY OF EQUATIONS

- If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the equation
  $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$
  where $a_0, a_1, \ldots, a_n$ are all real and $a_0 \neq 0$, then
  $$\sum \alpha_1 = -\frac{a_1}{a_0}, \ \sum \alpha_1\alpha_2 = +\frac{a_2}{a_0},$$
  $$\sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}, \ \alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

REMARKS

- If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of $f(x)$ and conversely.
- Every equation of n^{th} degree $(n \geq 1)$ has exactly n roots and if the equation has more than n roots, it is an identity.
- If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. *i.e.* imaginary roots occur in conjugate pairs.
- If the coefficients in the equation are all rational and $\alpha + \sqrt{\beta}$ is one of its root, then $\alpha - \sqrt{\beta}$

is also a root, where $\alpha, \beta \in Q$ and β is not a perfect square.

- If there be any two real numbers 'a' and 'b' such that $f(a)$ & $f(b)$ are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b' .
- Every equation $f(x) = 0$ of odd degree has atleast one real root of a sign opposite to that of its last term.

LOCATION OF ROOTS

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

- Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'd' are $b^2 - 4ac \geq 0$; $f(d) > 0$ and $(-b/2a) > d$.
- Conditions for both roots of $f(x) = 0$ to lie on either side of the number 'd' (in other words, the number 'd' lies between the roots of $f(x) = 0$) is $f(d) < 0$.
- Conditions for exactly one root of $f(x) = 0$ to lie in the interval (d, e) *i.e.* $d < x < e$ are $b^2 - 4ac > 0$ and $f(d) \cdot f(e) < 0$.
- Conditions that both roots of $f(x) = 0$ to be confined between the numbers p & q $(p < q)$ are $b^2 - 4ac \geq 0$; $f(p) > 0$; $f(q) > 0$ and $p < (-b/2a) < q$.

LOGARITHMIC INEQUALITIES

- For $a > 1$, the inequality $0 < x < y$ and $\log_a x < \log_a y$ are equivalent.
- For $0 < a < 1$, the inequality $0 < x < y$ and $\log_a x > \log_a y$ are equivalent.
- If $a > 1$, then $\log_a x < p \Rightarrow 0 < x < a^p$
- If $a > 1$, then $\log_a x > p \Rightarrow x > a^p$
- If $0 < a < 1$, then $\log_a x < p \Rightarrow x > a^p$
- If $0 < a < 1$, then $\log_a x > p \Rightarrow 0 < x < a^p$

PROBLEMS

SECTION-I

Single Correct Answer Type

1. All possible quadratic equations $ax^2 + bx + 1 = 0$ taking $a \in \{1, 2, 3, \ldots, n\}, b \in \{-1, -2, -3, \ldots, -n\}$ and if (α_i, β_i), $i = 1, 2, 3, \ldots n$, represent solution sets of these equations then $\sum \frac{1}{\alpha_i} + \sum \frac{1}{\beta_i}$ equals

(a) $\frac{n(n+1)}{2}$ (b) $\frac{n^2(n+1)}{2}$

(c) $\frac{n^3(n+1)}{2}$ (d) n^2

2. If $a \in (0, \pi)$ and if the equation $2x^3 - 6x + 2\sin a + 3 = 0$ has only one real root, then the maximum interval in which 'a' lies is

(a) $\left(0, \frac{\pi}{6}\right)$ (b) $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

(c) $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ (d) $\left(\frac{5\pi}{6}, \pi\right)$

3. If $f(x) = \prod_{i=1}^{3}(x - a_i) + \sum_{i=1}^{3}(a_i - x)$ where $a_i < a_{i+1}$, then $f(x) = 0$ has

(a) only one real root

(b) three real roots of which two of them are equal

(c) three distinct real roots

(d) three equal roots

4. Let $f(x) = x^2 - bx + c$, b is a odd positive integer, $f(x) = 0$ have two prime numbers as roots and $b + c = 35$. Then, the global minimum value of $f(x)$ is

(a) $-\frac{183}{4}$ (b) $\frac{173}{16}$

(c) $-\frac{81}{4}$ (d) data not sufficient

5. If $x^2 - ax + \cos^{-1}(x^2 - 6x + 10) + \sec^{-1}(6x - x^2 - 8) = 0$ has at least one real solution, then the number of possible integral values of 'a' in [2,8] is

(a) 4 (b) 3 (c) 2 (d) 1

6. Let α and β are the roots of $x^2 + 10x - 7 = 0$. If $a_n = \alpha^n + \beta^n$ for $n \geq 1$, then the value of $\frac{a_{12} - 7a_{10}}{2a_{11}}$ is

(a) 4 (b) -5 (c) 5 (d) -3

7. Given, $lx^2 - mx + 5 = 0$ does not have two distinct real roots, then the minimum value of $5l + m$ is

(a) 5 (b) -5 (c) 1 (d) -1

8. If t_n denote the n^{th} term of an A.P and $t_p = \frac{1}{q}$ and $t_q = \frac{1}{p}$, then which of the following is necessarily a root of the equation $(p + 2q - 3r)x^2 + (q + 2r - 3p)x + (r + 2p - 3q) = 0$

(a) t_p (b) t_q (c) t_{pq} (d) t_{p+q}

9. Number of integral values of x satisfying the inequality $\left(\frac{3}{4}\right)^{6x+10-x^2} < \frac{27}{64}$ is

(a) 6 (b) 7 (c) 8 (d) infinite

10. If the sum of the roots of the quadratic equation, $ax^2 + bx + c = 0$ is equal to sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

11. If the quadratic equations $3x^2 + ax + 1 = 0$ and $2x^2 + bx + 1 = 0$ have a common root, then the value of expression $|2a^2 - 5ab + 3b^2|$ is, where $(2a \neq 3b)$.

(a) 0 (b) 1 (c) 2 (d) 3

12. The number of values of 'p' for which the equation $(p^2 - 3p + 2)x^2 - (p^2 - 5p + 4)x + (p - p^2) = 0$ possesses more than two roots, is

(a) 0 (b) 1 (c) 2 (d) 4

13. If $(\alpha, \beta), (\beta, \gamma), (\gamma, \alpha)$ are respectively the roots of $x^2 - 2px + 2 = 0$; $x^2 - 2qx + 3 = 0$; $x^2 - 2rx + 6 = 0$ where α, β, γ are all positive, then the value of $p + q + r$ is

(a) 1 (b) 2 (c) 5 (d) 6

14. If one root of equation $x^2 - 4ax + a + f(a) = 0$ is three times of the other then minimum value of $f(a)$ is

(a) $-\frac{1}{6}$ (b) $-\frac{1}{10}$ (c) $-\frac{1}{5}$ (d) $-\frac{1}{12}$

15. A value of b for which the equations $x^2 + bx - 1 = 0$; $x^2 + x + b = 0$, have one root in common, is

(a) $-\sqrt{2}$ (b) $-i\sqrt{3}$ (c) $i\sqrt{5}$ (d) $\sqrt{2}$

16. Let α, β be the roots of the equation $(x - a)(x - b) = c, c \neq 0$, then the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are

(a) a, b (b) b, c
(c) a, c (d) $a + c, b + c$

17. The set of values of 'a' for which the equation $(a - 1)x^2 - (a + 1)x + (a - 1) \geq 0$ is true for all $x \geq 2$, is

(a) $\left(1, \frac{7}{3}\right]$ (b) $(-6, 1)$ (c) $\left[\frac{7}{3}, \infty\right)$ (d) $(-3, -2)$

18. a, b, c are all different and non zero real numbers which are in arithmetic progression. If the roots of the quadratic equation $ax^2 + bx + c = 0$ are α and β such that $\frac{1}{\alpha} + \frac{1}{\beta}, \alpha + \beta, \alpha^2 + \beta^2$ are in geometric progression, then the value of $\frac{a}{c}$ is

(a) 1 (b) 2 (c) 3 (d) 4

19. If $a \in R$ and the equation $(a-2)(x-[x])^2 + 2(x-[x]) + a^2 = 0$ (where $[x]$ denotes G.I.F) has no integral solution and has exactly one solution in (2, 3) then a lies in the interval

(a) (0, 1) (b) (1, 2) (c) (–1, 3) (d) (–1, 0)

20. Consider the quadratic equation $ax^2 - bx + c = 0$, $a, b, c \in N$. If the given equation has two distinct real roots belonging to (1, 2), then

(a) $1 < a < 5$ (b) $a \geq 5$
(c) $a = 4$ (d) $a = 3$

21. The sum of all the real roots of the equation $|x - 2|^2 + |x - 2| - 2 = 0$ is

(a) 1 (b) 2 (c) 3 (d) 4

22. Let α be a root of $ax^2 + bx + c = 0$ and β be a root of $-ax^2 + bx + c = 0$, where a, b and c are real numbers and $a \neq 0$. Then the equation $\frac{a}{2}x^2 + bx + c = 0$ has a root γ such that

(a) $\gamma < \min\{\alpha, \beta\}$
(b) $\gamma > \max\{\alpha, \beta\}$
(c) γ lies between α and β
(d) $-\gamma$ lies between α and β

23. If a, b, c and d are distinct positive real numbers such that a and b are the roots of $x^2 - 10cx - 11d = 0$ and c and d are the roots of $x^2 - 10ax - 11b = 0$, then the value of $a + b + c + d$ is

(a) 1110 (b) 1010
(c) 1101 (d) 1210

24. The value of a for which exactly one root of the equation $e^a x^2 - e^{2a}x + e^a - 1 = 0$ lies between 1 and 2 are given by

(a) $\left(\log_e \frac{4-\sqrt{13}}{4}, \log_e \frac{4+\sqrt{13}}{2}\right)$

(b) $\left(0, \log_e \frac{4+\sqrt{13}}{2}\right)$

(c) $\left(\log_e \frac{5-\sqrt{17}}{4}, \log_e \frac{5+\sqrt{17}}{4}\right)$

(d) $\left(\log_e \frac{6-\sqrt{17}}{2}, \log_e \frac{6+\sqrt{17}}{2}\right)$

25. If $0 \le a \le 3$, $0 \le b \le 3$ and the equation $x^2 + 4 + 3\cos(ax + b) = 2x$ has at least one solution then $(a + b)$ is

(a) 0 (b) $\pi/2$ (c) π (d) 3π

26. If the roots of $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, then a, b, c are in

(a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

27. Number of positive integral solution (x, y) of the equation $2xy - 4x^2 + 12x - 5y = 11$ is

(a) 0
(b) 1
(c) 2
(d) infinitely many solutions

28. Number of ordered pair (x, y) satisfying the condition $x^8 + y^8 = 8xy - 6$ is

(a) 2
(b) 4
(c) 16
(d) infinitely many solutions

29. The equation $x^2 + bx + c = 0$ has distinct roots. If 2 is subtracted from each root, the result are the reciprocal of the original roots, then $b^2 + c^2$ is

(a) 2 (b) 3 (c) 4 (d) 5

SECTION-II

Multiple Correct Answer Type

30. The set S of all real "x" for which $(x^2 - x + 1)^{x-1} < 1$ contains

(a) $(-5, -1)$ (b) $(-1, 1)$
(c) $(-1, 0)$ (d) $(-3, 1)$

31. For real x, the function $\frac{(x-a)(x-b)}{x-c}$ will assume all real values provided

(a) $a > b > c$ (b) $a < b < c$
(c) $a > c > b$ (d) $a < c < b$

32. Given, $ax^2 + bx + c \ge 0$, $bx^2 + cx + a \ge 0$ and $cx^2 + ax + b \ge 0$, where $a \ne b \ne c$ and $a, b, c \in R$. Now $\frac{a^2+b^2+c^2}{ab+bc+ca}$ cannot take values

(a) $\frac{7}{9}$ (b) $\frac{2}{7}$ (c) $\frac{16}{3}$ (d) $\frac{3}{2}$

33. If the quadratic equation $ax^2 + bx + c = 0$ $(a > 0)$ has $\sec^2\theta$ and $\operatorname{cosec}^2\theta$ as its roots, then which of the following hold good?

(a) $b + c = 0$ (b) $b^2 - 4ac \ge 0$
(c) $c \ge 4a$ (d) $4a + b \ge 0$

34. If the equation $x^2 + bx - a = 0$ and $x^2 - ax + b = 0$ have a common root, then

(a) $a + b = 0$ (b) $a = b$
(c) $a - b = 1$ (d) $a < \frac{1}{4}$

35. If a, b, c are in G.P., where a, c are positive, then the equation $ax^2 + bx + c = 0$ has

(a) real roots (b) imaginary roots
(c) ratio of roots = 1 : ω, ω is non-real cube root of unit
(d) ratio of roots = $b : ac$

36. If the real numbers a, b, c are distinct, then the roots of the equation

$(c^2 - ab)x^2 - 2(a^2 - ac)x + (b^2 - ac) = 0$

will be equal, if

(a) $a = 0$
(b) $a + b + c = 0$
(c) $a^3 + b^3 + c^3 - 3abc = 0$
(d) $a^2 + b^2 + c^2 - bc - ca - ab = 0$

37. If $x^2 - 2x + \sin^2\alpha = 0$, then

(a) $x \in [-1, 1]$ (b) $x \in [0, 2]$
(c) $x \in [1, 2]$ (d) $x \in [-2, 2]$

38. If the difference of the roots of the equation $x^2 + hx + 7 = 0$ is 6, then possible value (s) of 'h' are

(a) -4 (b) 4
(c) -8 (d) 8

39. If $\cos\alpha$ is a root of the equation $169x^2 - 26x - 35 = 0$, $-1 < x < 0$, then $\sin 2\alpha$ is

(a) $\frac{144}{169}$ (b) $-\frac{144}{169}$ (c) $\frac{120}{169}$ (d) $-\frac{120}{169}$

40. Let $a, b, c \in Q^+$ satisfying $a > b > c$. Which of the following statements (s) hold true for the quadratic polynomial $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b)$?

(a) The mouth of the parabola $y = f(x)$ opens upwards.
(b) Both roots of the equation $f(x) = 0$ are rational.
(c) x-coordinate of vertex of the graph is positive
(d) x-coordinate of vertex of the graph is negative

41. Let a, b and c be real numbers and $a \neq 0$. Let α and β be the roots of $ax^2 + bx + c = 0$. If α' and β' are roots of the equation $a^3x^2 + (abc)x + c^3 = 0$, then

(a) $\alpha' = \alpha^3\beta^2$ (b) $\beta' = \beta^3\alpha^2$
(c) $\alpha' = \alpha^2\beta$ (d) $\beta' = \alpha\beta^2$

42. If $c \in R$ and the negative of one of the solutions of $x^2 - 3x + c$ is a solution of $x^2 + 3x - c = 0$, then the solutions of $x^2 - 3x + c = 0$ are

(a) -1 (b) 1 (c) 0 (d) 3

43. The equation $(ay - bx)^2 + 4xy = 0$ has rational solutions x, y for

(a) $a = 1/2, b = 2$ (b) $a = 4, b = 1/8$
(c) $a = 1, b = 3/4$ (d) $a = 2, b = 1$

44. If one root of $kx^2 + x + 1 = 0$ is real and other is imaginary ($k \neq 0$), then k^2 can not be equal to

(a) -4 (b) -1 (c) 1 (d) 4

45. Let a, b, c be three distinct non-zero real numbers satisfying the system of equation: $\frac{1}{a}+\frac{1}{a-1}+\frac{1}{a-2}=1$; $\frac{1}{b}+\frac{1}{b-1}+\frac{1}{b-2}=1$; $\frac{1}{c}+\frac{1}{c-1}+\frac{1}{c-2}=1$ then

(a) $a + b + c = 6$ (b) $abc = 2$
(c) $(1 - a)(1 - b)(1 - c) = 1$
(d) $(a - 2)(b - 2)(c - 2) = 2$

46. The real values of 'a' for which the equation $x^4 - 2x^2a - x^2 + 6x + a^2 - 3a = 0$ has all its roots real

(a) $a > 4$ (b) $a \geq 2$
(c) $a \geq 3/4$ (d) $a \geq -1$

47. Let 'm' be a real number, and suppose that two of the three solutions of the cubic equation $x^3 + 3x^2 - 34x = m$ differ by 1. Then possible values of 'm' is /are

(a) 120 (b) 80 (c) -48 (d) -32

48. If both the roots of the equation $x^2 - 2ax + a^2 + a - 3$ in the variable x are less than 3, then a can be

(a) 2 (b) 5/2 (c) $\sqrt{3}$ (d) -7

49. If the equation $x^2 + (a - b)x - a - b + 1 = 0$, where $a, b \in R$ has unequal real roots, then

(a) $a > 0$ (b) $a > 1$ (c) $a > -1$ (d) $a > 2$

50. The inequality $\left|\frac{x^2+kx+1}{x^2+x+1}\right| < 3$ is satisfied for all real values of x, then

(a) $k \in (-1, 5)$
(b) The number of integral values of k is 5
(c) $k \in (-5, 1)$
(d) The equation $x^2 - (k^2 - 4k - 5)x - 1 = 0$ has one real root in $(0, 1)$

SECTION-III

Comprehension Type

Paragraph for Question No. 51 to 53

Let $f(x) = x^2 + b_1x + c_1$, $g(x) = x^2 + b_2x + c_2$. Let the real roots of $f(x) = 0$ be α, β and real roots of $g(x) = 0$ be $\alpha + h$, $\beta + h$. The least value of $f(x)$ is $-1/4$ and least value of $g(x)$ occurs at $x = 7/2$ then

51. The least value of $g(x)$ is

(a) $-1/4$ (b) -1 (c) $-1/3$ (d) $-1/2$

52. The value of b_2 is

(a) -5 (b) 9 (c) -8 (d) -7

53. The roots of $f(x) = 0$ are

(a) $3, -4$ (b) $-3, 4$ (c) $3, 4$ (d) $-3, -4$

Paragraph for Question No. 54 to 56

Let $f(x) = ax^2 + bx + c$, where a, b and c are real and $a \neq 0$. Let $\alpha < \beta$ be the roots of $f(x) = 0$. Then

(A) for all x such that $\alpha < x < \beta$, $f(x)$ and a have opposite signs

(B) for $x < \alpha$ or $x > \beta$, $f(x)$ and a have the same sign.

54. If both the roots of the equation $x^2 - mx + 1 = 0$ are less than unity, then

(a) $m \le -2$ (b) $m > 2$
(c) $-1 \le m \le 3$ (d) $0 \le m \le 5/2$

55. If both the roots of the equation $x^2 - 6mx + 9m^2 - 2m + 2 = 0$ are greater than 3, then
(a) $m < 0$ (b) $m > 1$
(c) $0 < m < 1$ (d) $m > 11/9$

56. If both the roots of the equation $4x^2 - 2x + m = 0$ belong to the interval $(-1, 1)$, then
(a) $-2 \le m \le \frac{3}{4}$ (b) $0 < m < 2$
(c) $2 < m < \frac{5}{2}$ (d) $-2 < m \le \frac{1}{4}$

Paragraph for Question No. 57 to 59

Let m_1 and m_2 are two integral values of 'm' $(m_1 < m_2)$ for which roots of $x^2 + 19x + 92 - m^2 = 0$ are also integers. Now a quadratic expression $f(x) = x^2 + (a^2 + 1)x + 2x + 2(a^2 + 1)$ is such that it is negative for exactly two negative integral values of x.

57. The value of $m_1^2 + m_2^2$ is given by
(a) 4 (b) 8 (c) 12 (d) 18

58. a^2 belongs to
(a) (3, 4) (b) [3, 4] (c) (3, 4] (d) [1, 2]

59. The value of $f(m_2)$ is
(a) always positive (b) always negative
(c) depends upon a (d) nothing can be said

Paragraph for Question No. 60 to 62

a, b, c, d are real numbers, not necessarily pairwise distinct. In this paragraph, we wish to find all the four tuples (a, b, c, d), such that each of the expressions $abc + d$, $bcd + a$, $cda + b$ and $dab + c$, equals 2.

60. The quadratic equation $Px^2 - 2x + Q = 0$ is satisfied by a, b, c, d. Hence
(a) $P = abcd$ (b) $Q = 2$
(c) $P = 1$ (d) $Q = 1$

61. If a, b, c, d are not all equal, then $a + b + c + d$ equals
(a) 1 (b) 0 (c) -1 (d) 2

62. The number of four tuples (a, b, c, d) (counting all arrangements), satisfying the hypothesis, is
(a) 2 (b) 8 (c) 5 (d) 6

Paragraph for Question No. 63 to 65

Let x_0, x_1 and x_2 be three distinct real numbers. Define three polynomials $l_0(x)$, $l_1(x)$, $l_2(x)$ and $l(x)$ as follows :

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)};$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)};$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$l(x) = (x - x_0)(x - x_1)(x - x_2)$$

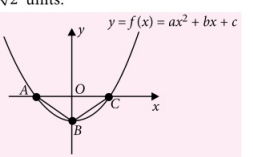
63. $P(x) = (x_1 + x_2)\, l_0(x) + (x_2 + x_0)\, l_1(x) + (x_0 + x_1)\, l_2(x)$ then $P(x)$ equals
(a) x (b) $x - \frac{1}{2}(x_0 + x_1 + x_2)$
(c) $x_0 + x_1 + x_2 - x$ (d) $\frac{x_0 + x_1 + x_2}{3}$

64. $\frac{l(x)}{(x - x_0)l'(x_0)} + \frac{l(x)}{(x - x_1)l'(x_1)} + \frac{l(x)}{(x - x_2)l'(x_2)}$ equals
(a) 1 (b) x (c) $x^2 + x$ (d) $x - 1$

65. $\sum_{k=0}^{2} l_k(x)x^2$ equals
(a) 0 (b) x (c) x^2 (d) x^3

Paragraph for Question No. 66 to 68

In the given figure, vertices of ΔABC lie on $y = f(x) = ax^2 + bx + c$. The ΔABC is right angled isosceles triangle, whose hypotenuse $AC = 4\sqrt{2}$ units.



66. $y = f(x)$ is given by
(a) $y = x^2 - 2\sqrt{2}$ (b) $y = x^2 - 12$
(c) $y = \frac{x^2}{2} - 2$ (d) $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

67. The minimum value of $y = f(x)$ is

(a) -4 (b) -2 (c) $-2\sqrt{2}$ (d) -12

68. Number of integral values of 'K' for which one root of $f(x) = 0$ is more than 'K' and other less than 'K' is

(a) 4 (b) 5 (c) 6 (d) 7

SECTION-IV

Matrix-Match Type

69. Match the following

	Column I		Column II
(A)	If inequation $ax^2 - ax + 1 > 0$ $\forall\ x \in R$ then a belongs to	(p)	$(-4, 4)$
(B)	If $x^3 - 3x + \frac{a}{2} = 0$ has three real and distinct roots, then a belongs to	(q)	$(0, 4)$
(C)	If $x^3 + ax^2 + x + 1 = 0$ is an increasing function, then a may belong to	(r)	$(-3, 3)$
(D)	If quadratic equation $x^2 - 3ax + a^2 - 9 = 0$ has roots of opposite sign then a belongs to	(s)	$[-\sqrt{3}, \sqrt{3}]$

70. Match the equations on the left with the properties on the right.

	Column I		Column II
(A)	$a, b, c \in R$ are such that $a + b + c = 0$, equation is $3ax^2 + 2bx + c = 0$	(p)	real roots
(B)	$a, b, c \in R$ are such that $2a + 3b + 6c = 0$ equation is $ax^2 + bx + c = 0$	(q)	discriminant > 0
(C)	$a, b, c \in R$ are such that $(a + c)^2 < b^2$, equation is $ax^2 + bx + c = 0$	(r)	at least one root in $(0, 1)$
(D)	$a, b, c \in R$ are such that $1 + \frac{\lvert b \rvert}{a} + \frac{c}{a} < 0$	(s)	a root in $(-\infty, -1)$ or a root in $(1, \infty)$

71. Match the following for the equation $x^2 + a|x| + 1 = 0$, where a is a parameter

	Column I		Column II
(A)	No real root	(p)	$a < -2$
(B)	Two real roots	(q)	$a = -2$
(C)	Three real roots	(r)	ϕ
(D)	Four real roots	(s)	$a \geq 0$

72. Match the items in Column I with those in Column-II

	Column I		Column II
(A)	The values of k for which both the roots of the equation $x^2 - 6kx + 2 - 2k + 9k^2 = 0$ are greater than 3 belong to	(p)	$(2, 5)$
(B)	If $\log_{0.1}(x^2 + x) > \log_{0.5}(x^3 - x) + \log_2(x - 1)$, then x belongs to	(q)	$\left(-\infty, \frac{3}{2}\right)$
(C)	If $\lvert x - 1 - x^2 \rvert \leq \lvert x^2 - 3x + 4 \rvert$, then x belongs to	(r)	$(1, +\infty)$
(D)	If $\lvert x^2 - 2x - 3 \rvert < 3x - 3$, then x lies in the interval	(s)	$\left(\frac{11}{9}, +\infty\right)$
		(t)	$[2, 5]$

73. Match the following:

	Column I		Column II
(A)	If all the roots of $x^3 + ax^2 + bx + c$ are positive integers greater than 2, and the coefficients satisfy $a + b + c + 1 = -2009$ then $\lvert a \rvert =$	(p)	4
(B)	Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial, where a, b, c, d are integers. Suppose $P(0) = 2006$, $P(1) = 2007$, $P(2) = 2008$. What is the smallest possible positive value of $P(3)$?	(q)	6
(C)	Let $A(\alpha)$, $B(\beta)$ be distinct complex numbers such that $\frac{\alpha}{\beta} + \frac{\bar{\alpha}}{\bar{\beta}} = 2$ and $\alpha \neq 0$, $\beta \neq 0$ then $\frac{8}{\pi} \angle OBA =$	(r)	5

(D)	Let $r_1, r_2, ..., r_n$ be n positive integers, not necessarily distinct, such that $(x + r_1)(x + r_2) ... (x + r_n) = x^n + 56x^{n-1} + ... +$ 2009, the value of n is	(s)	58

74. Consider the equation in real numbers $|x - 3| + |x + 9| = a$, $a \in R$. Match the statements/ expressions in column I with those in column II.

Column I		Column II	
(A)	As a range over R, the maximum possible number of integral solutions is	(p)	2
(B)	For $a > \log_{27}81$, the number of solutions cannot be	(q)	0
(C)	For $a = 13$, the number of solutions is	(r)	13
(D)	For at least one solution 'a' cannot take the value(s)	(s)	1

SECTION-V

Integer Answer Type

75. Number of values of x satisfy the equation $|x^2 + 4x + 3| + 2x + 5 = 0$ (is/are)

76. Suppose a cubic polynomial $f(x) = x^3 + px^2 + qx + 72$ is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$ (where a, b, p, q are constants and $a \neq b$), then the value of p is

77. If both the roots of the equation $x^2 - 2ax + (a^2 + a - 3) = 0$ are less than 3 then maximum value of $[a]$ is (where $[x]$ denotes the greatest integer function)

78. If $[x]$ denotes the greatest integer less than or equal to x then the number of integer values of x which satisfies the equation
$[x^2 - 2]^2 - 9[x^2 - 2] + 14 = 0$ is

79. Let $f(x) = ax^2 + bx + c$ (where $a \neq 0, a, b, c \in R$) and $(a + c)^2 < b^2$, then the number of distinct values of x in $(-1, 1)$ satisfying the equation $f(x) = 0$ will be

80. The total number of integral values of a so that $x^2 - (a + 1)x + a - 1$ has integral roots, is equal to

81. If $x^2 + 3x + 5 = 0$ and $ax^2 + bx + 2 = 0$ have a common root and $a, b \in R$, then the value of $a + b$ is equal to $k/5$, then $k =$

82. All the values of parameter m, for which $x^2 + 2(m - 1)x + m + 5$ is positive for $x > 1$, lie in the interval $(-k, \infty)$ then k is

83. If $f(x)$ is continuous function and attains only rational values. If $f(0) = 3$, then number of different solutions of the equation $f(1)x^2 + 2f(2)x + f(3) = 0$ are

84. If the equation $x^2 + 2\alpha x + \alpha^2 - 1 = 0$ and $x^2 + 2\beta x + \beta^2 - 1 = 0$ have a common root ($\alpha \neq \beta$), then the value of the expression $2\alpha^2 - 4\alpha\beta - |\alpha - \beta| + |\alpha - \beta|\beta^2$, is

85. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

86. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots is

SOLUTIONS

1. (b) : $ax^2 + bx + 1 = 0$ has roots α_i, β_i

$x^2 + bx + a$ has roots $\frac{1}{\alpha_i}, \frac{1}{\beta_i}$. There are n^2 equations

Given expression $= \sum \frac{1}{\alpha_i} + \sum \frac{1}{\beta_i} = \sum(-b)$

$= n(1 + 2 + 3 + ... + n) = \frac{n \cdot n(n+1)}{2} = \frac{n^2(n+1)}{2}$

2. (c) : Let $f(x) = 2x^3 - 6x + 2\sin a + 3$
$\Rightarrow f'(x) = 6x^2 - 6 = 0 \Rightarrow x = \pm 1$
$f(x)$ has a local maximum at -1 and local minimum at 1.
For the function $f(x) = 0$, to have only one real root $f(-1) \cdot f(1) > 0$
$\Rightarrow (2\sin a + 7)(2\sin a - 1) > 0$

$\Rightarrow \sin a > \frac{1}{2} (\because 2\sin a + 7 > 0 \ \forall a \in R)$

$\Rightarrow a \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

3. (c) : $f(x) = (x - a_1)(x - a_2)(x - a_3) + (a_1 - x) + (a_2 - x) + (a_3 - x)$
Now, $f(x) \to -\infty$ as $x \to -\infty$ and $f(x) \to \infty$ as $x \to \infty$.

Again, $f(a_1) = (a_2 - a_1) + (a_3 - a_1) > 0$
$[\because a_1 < a_2 < a_3]$
$\Rightarrow$ One root belongs to $(-\infty, a_1)$
Also, $f(a_3) = (a_1 - a_3) + (a_2 - a_3) < 0$
$\Rightarrow$ One root belongs to (a_1, a_3)
So, $f(x) = 0$ has three distinct real roots.

4. (c) : Let a, b be roots of $x^2 - bx + c = 0$,
Then, $a + b = b$
$\Rightarrow$ one of the roots is '2' (Since a, b are primes and b is odd positive integer)
$\Rightarrow f(2) = 0 \Rightarrow 2b - c = 4$ and $b + c = 35$
$\Rightarrow b = 13, c = 22$
Minimum value $= f\left(\frac{13}{2}\right) = -\frac{81}{4}$

5. (d) : Valid only for $x = 3$
$\Rightarrow 3a = 9 \Rightarrow a = 3$

6. (b) : $x^2 + 10x - 7 = 0$
$x^{12} + 10x^{11} - 7x^{10} = 0$
$\left.\begin{aligned}\alpha^{12} + 10\alpha^{11} - 7\alpha^{10} = 0\\ \beta^{12} + 10\beta^{11} - 7\beta^{10} = 0\end{aligned}\right\}$ adding, we get
$(\alpha^{12} + \beta^{12}) + 10(\alpha^{11} + \beta^{11}) - 7(\alpha^{10} + \beta^{10}) = 0$
or, $a_{12} + 10a_{11} - 7a_{10} = 0$
or, $a_{12} - 7a_{10} = -10a_{11}$ or $\frac{a_{12} - 7a_{10}}{2a_{11}} = -5$

7. (d) : $f(x) = lx^2 - mx + 5$
$f(0) = 5 > 0 \Rightarrow f(x) \geq 0 \ \forall \ x \in R$
$\Rightarrow f(-5) > 0$
$\Rightarrow 25l + 5m + 5 \geq 0$ or $5l + m \geq -1$

8. (c) : Clearly $x = 1$ is a root of given equation. Let a be the first term and d be the common difference.

$$\left.\begin{aligned}t_p = a + (p-1)d = \frac{1}{q}\\ t_q = a + (q-1)d = \frac{1}{p}\end{aligned}\right\} \text{ solving } a = d = \frac{1}{pq}$$

$\therefore \ t_{pq} = a + (pq - 1)d = 1$
$\therefore \ t_{pq}$ is the root of the given equation.

9. (b) : $6x + 10 - x^2 > 3$
or $x^2 - 6x - 7 < 0$ or $(x + 1)(x - 7) < 0$

10. (c) : Let the roots of the equation $ax^2 + bx + c = 0$ are α and β

$$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2}$$

or $(\alpha + \beta)(\alpha\beta)^2 = (\alpha + \beta)^2 - 2\alpha\beta$

or $\left(-\frac{b}{a}\right)\left(\frac{c^2}{a^2}\right) = \frac{b^2}{a^2} - \frac{2c}{a}$

or $-bc^2 = ab^2 - 2a^2c$

or $ab^2 + bc^2 = 2a^2c$

Dividing by abc

$$\frac{b}{c} + \frac{c}{a} = \frac{2a}{b}$$

$\Rightarrow \frac{b}{c}, \frac{a}{b}, \frac{c}{a}$ are in A.P.

$\Rightarrow \frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in H.P.

11. (b) : $3x^2 + ax + 1 = 0 \Rightarrow 6x^2 + 2ax + 2 = 0$
$2x^2 + bx + 1 = 0 \Rightarrow 6x^2 + 3bx + 3 = 0$
$\Rightarrow (2a - 3b)x - 1 = 0$
or $x = \frac{1}{2a - 3b}$, substitute in any equation, we get
$2 + b(2a - 3b) + (2a - 3b)^2 = 0$
$\Rightarrow 2 + 2ab - 3b^2 + 4a^2 + 9b^2 - 12ab = 0$
$\Rightarrow 4a^2 + 6b^2 - 10ab + 2 = 0$
$\Rightarrow |2a^2 - 5ab + 3b^2| = |-1| = 1$

12. (b) : Since given equation possess more than two roots
$\therefore$ it is an identity
$\therefore (p^2 - 3p + 2) = 0 \Rightarrow p = 1, 2$
$(p^2 - 5p + 4) = 0 \Rightarrow p = 1, 4$
$(p - p^2) = 0 \Rightarrow p = 0, 1$

13. (d) : $\alpha\beta = 2$, $\beta\gamma = 3$, $\gamma\alpha = 6$
$(\alpha\beta\gamma)^2 = 36 \Rightarrow (\alpha\beta\gamma) = 6 \Rightarrow \alpha = 2, \beta = 1, \gamma = 3$

14. (d) : Roots are α, 3α
$\Rightarrow 4\alpha = 4a \Rightarrow \alpha = a$
$a^2 - 4a^2 + a + f(a) = 0 \Rightarrow f(a) = 3a^2 - a$
$\therefore$ Min. value $= \frac{4 \cdot 3 \cdot 0 - (-1)^2}{4 \cdot 3} = -\frac{1}{12}$

15. (b) : Using condition for common root
$(c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$
$\Rightarrow (-1 - b)^2 = (1 - b)(b^2 + 1)$
$\Rightarrow b^3 + 3b = 0 \Rightarrow b = 0, \pm i\sqrt{3}$

16. (a) : α, β are roots of the equation $(x - a)(x - b) = c$, $c \neq 0$
$\therefore (x - a)(x - b) - c = (x - \alpha)(x - \beta)$
$\Rightarrow (x - a)(x - b) + c = (x - \alpha)(x - \beta)$
$\Rightarrow$ roots of $(x - \alpha)(x - \beta) + c = 0$ are also a and b.

17. (a) : Given, $(a-1)x^2-(a+1)x+(a-1)\geq 0$

$\Rightarrow a(x^2-x+1)-(x^2+x+1)\geq 0$

or $a\geq \dfrac{x^2+x+1}{x^2-x+1}=1+\dfrac{2x}{x^2-x+1}=1+\dfrac{2}{x+\frac{1}{x}-1}$

$\Rightarrow a\in\left(1,\dfrac{7}{3}\right]$

18. (c) : Since a, b, c are in A.P. $\therefore 2b=a+c$ ...(i)

and since $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$, $\alpha+\beta$, $\alpha^2+\beta^2$ are in G.P.

$\therefore (\alpha+\beta)^2=\left(\dfrac{\alpha+\beta}{\alpha\beta}\right)(\alpha^2+\beta^2)$

$\Rightarrow \dfrac{b^2}{a^2}=\dfrac{-b}{c}\left(\dfrac{b^2}{a^2}-2.\dfrac{c}{a}\right)$

$\Rightarrow bc=-b^2+2ac$ or $b(b+c)=2ac$

$\left(\dfrac{a+c}{2}\right)\left(\dfrac{a+3c}{2}\right)=2ac$ (From (i))

or $a^2+3c^2+4ac=8ac$

or $\left(\dfrac{a}{c}\right)^2-4\left(\dfrac{a}{c}\right)+3=0$ or $\dfrac{a}{c}=1;\dfrac{a}{c}=3$

Since a, b, c are all different

$\therefore \dfrac{a}{c}=3$ is true.

19. (d) : $(a-2)(x-[x])^2+2(x-[x])+a^2=0$ ... (1)

$f(y)=(a-2)y^2+2y+a^2=0$,

where $y=x-[x]$ ... (2)

Since x cannot be an integer

when $2<x<3 \Rightarrow [x]=2$ and $0<y<1$

$\therefore$ Equation (1) has exactly one solution in (2, 3)

$\therefore$ Equation (2) has exactly one solution in (0, 1)

$\therefore f(0)\cdot f(1)<0$

$\Rightarrow a^2(a-2+2+a^2)<0$

$\Rightarrow a(a+1)<0$

$\Rightarrow a\in(-1,0)$

20. (b) : $\alpha-1, \beta-1, 2-\alpha, 2-\beta\in(0,1)$

Use A.M. $\geq$ G.M., we get

$\dfrac{(\alpha-1)+(2-\alpha)}{2}\geq\sqrt{(\alpha-1)(2-\alpha)}$

$\Rightarrow (\alpha-1)(2-\alpha)<\dfrac{1}{4}, (\beta-1)(2-\beta)<\dfrac{1}{4}$

$\Rightarrow 0<(\alpha-1)(\beta-1)(2-\alpha)(2-\beta)<\dfrac{1}{16}$

$\Rightarrow 0<\left(\dfrac{c}{a}-\dfrac{b}{a}+1\right)\left(4-\dfrac{2b}{a}+\dfrac{c}{a}\right)<\dfrac{1}{16}$

$\Rightarrow 0<(a-b+c)(4a-2b+c)<\dfrac{a^2}{16}$

$(a-b+c)(4a-2b+c)\geq 1$

$\Rightarrow \dfrac{a^2}{16}>1\Rightarrow a>4 \Rightarrow a\geq 5$

21. (d) : Case 1 : Suppose $x\geq 2$. Then the equation becomes

$(x-2)^2+(x-2)-2=0$

or $x^2-3x=0$

Since $x\geq 2$, we get $x=3$.

Case 2 : Suppose $x<2$. Then the equation becomes

$(2-x)^2+(2-x)-2=0$

$x^2-5x+4=0$ or $(x-1)(x-4)=0$

But $x<2$. Therefore $x=1$. Thus the real roots of the equation are 1 and 3 and their sum is 4.

22. (c) : By hypothesis, $a\alpha^2+b\alpha+c=0$ ...(i) and $a\beta^2-b\beta-c=0$ ...(ii)

Let $f(x)=\dfrac{a}{2}x^2+bx+c$

Then $f(\alpha)=\dfrac{a}{2}\alpha^2+b\alpha+c=\dfrac{1}{2}(a\alpha^2+2b\alpha+2c)$

$=\dfrac{1}{2}(a\alpha^2-2a\alpha^2)=-\dfrac{a}{2}\alpha^2$ (from (i))

and $f(\beta)=\dfrac{a}{2}\beta^2+b\beta+c=\dfrac{1}{2}(a\beta^2+2b\beta+2c)$

$=\dfrac{1}{2}(a\beta^2+2a\beta^2)=\dfrac{3}{2}a\beta^2$ (from (ii))

Therefore, $f(\alpha)f(\beta)=\dfrac{-3a^2\alpha^2\beta^2}{4}<0$

Hence $f(x)=0$ has a root lie in between α and β.

23. (d) : Since a and b are the roots of $x^2-10cx-11d=0$, we have

(i) $a+b=10c$ and (ii) $ab=-11d$ ... (1)

Also, since c and d are the roots of $x^2-10ax-11b=0$, we have

(i) $c+d=10a$ and (ii) $cd=-11b$ ... (2)

Adding part (i) of Eqs. (1) and (2), we get

$a+b+c+d=10(a+c)$

$\Rightarrow b+d=9(a+c)$ ... (3)

Multiplying (ii) of Eqs. (1) and (2), we get
$abcd = 121bd \Rightarrow ac = 121$ ... (4)
Also, $a^2 - 10ca - 11d = 0 = c^2 - 10ca - 11b$
$\Rightarrow a^2 + c^2 - 20ca - 11(b + d) = 0$
From Eqs. (3) and (4), we have
$a^2 + c^2 - 20(121) - 99(a + c) = 0$
$\Rightarrow (a + c)^2 - 2 \times 121 - 20 \times 121 - 99(a + c) = 0$
$\Rightarrow (a + c - 121)(a + c + 22) = 0$
$\Rightarrow a + c = 121 \; ; \; a + c = -22$
Since a, c are positive, $a + c \neq -22$. Therefore $a + c = 121$ and
$a + b + c + d = (a + c) + 9(a + c) = 1210$

24. (c) : $f(1)\, f(2) < 0$
$\Rightarrow (e^a - e^{2a} + e^a - 1)(4e^a - 2e^{2a} + e^a - 1) < 0$
Let $e^a = y \Rightarrow \log y = a$
$\Rightarrow (y^2 - 2y + 1)(2y^2 - 5y + 1) < 0$
or $(y - 1)^2 (2y^2 - 5y + 1) < 0$
or $2y^2 - 5y + 1 < 0$
or $\dfrac{5-\sqrt{17}}{4} < y < \dfrac{5+\sqrt{17}}{4}$

$$\log_e \frac{5-\sqrt{17}}{4} < a < \log_e \frac{5+\sqrt{17}}{4}$$

25. (c) : $(x - 1)^2 + 3 = -3\cos(ax + b)$
Range of L.H.S $= [3, \infty)$
Range of R.H.S $= [-3, 3]$
$\therefore \; -3\cos(a + b) = 3$
$\Rightarrow a + b = \pi$

26. (c) : $x = 1$ is a root $\Rightarrow$ other root is also 1
$\therefore$ product of roots $= 1$

$$\Rightarrow \frac{c(a-b)}{a(b-c)} = 1 \Rightarrow 2ac = ab + bc \Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

27. (c) : $y = 2x - 1 + \dfrac{6}{2x-5}$

for y to be an integer, $\dfrac{6}{2x-5}$ must be an integer
$\Rightarrow 2x - 5 = \pm 1, \pm 3, \pm 6$ but $\pm 3, \pm 6$ are not possible.
$\Rightarrow$ solutions are (3, 11), (4, 9)

28. (b) : $x^8 + y^8 + 6 = 8xy$
Applying A.M. $\geq$ G.M.

$$\frac{x^8 + y^8 + 1 + 1 + 1 + 1 + 1 + 1}{8} \geq (x^8 y^8)^{1/8}$$

$\Rightarrow x^8 + y^8 + 6 \geq 8xy$
$\Rightarrow x^8 = y^8 = 1 \Rightarrow x = \pm 1, y = \pm 1$

29. (d) : Let r_1, r_2 be the roots, from given condition

$$r - 2 = \frac{1}{r} \Rightarrow r^2 - 2r - 1 = 0,$$

$b = -2$, $c = -1$, $b^2 + c^2 = 5$
(comparing with $x^2 + bx + c = 0$)

30. (a, c) : $(x^2 - x + 1)^{x-1} < 1$
$\Rightarrow (x - 1)\log(x^2 - x + 1) < 0$
Case (i) : $x - 1 > 0$, $\log(x^2 - x + 1) < 0$
$x \in (0, 1) \Rightarrow$ no solution
Case (ii) : $x - 1 < 0$ or $x < 1$, $\log(x^2 - x + 1) > 0$
$x \in (-\infty, 0) \cup (1, \infty)$

31. (c, d) : Let $y = \dfrac{(x-a)(x-b)}{(x-c)}$
$\Rightarrow (x - c)y = x^2 - (a + b)x + ab$
$\Rightarrow x^2 - (a + b + y)x + ab + cy = 0$
Here, $\Delta = (a + b + y)^2 - 4(ab + cy)$
$= y^2 + 2y(a + b - 2c) + (a - b)^2$
Since x is real and y assumes all real values,
$\therefore \; \Delta \geq 0$ for all real values of y
$\Rightarrow y^2 + 2y(a + b - 2c) + (a - b)^2 \geq 0$
Now, we know that the sign of a quadratic is same as that of coeff of y^2 provided its discriminant
$B^2 - 4AC < 0$
This will be so if,
$4(a + b - 2c)^2 - 4(a - b)^2 < 0$
or $4(a + b - 2c + a - b)$
$(a + b - 2c - a + b) < 0$
$\Rightarrow 16(a - c)(b - c) < 0$
$\Rightarrow 16(c - a)(c - b) < 0$ ... (1)
Now, If $a < b$ then from inequation (1), we get
$c \in (a, b) \Rightarrow a < c < b$
or If $a > b$ then from inequation (1), we get $c \in (b, a)$
$\Rightarrow b < c < a$ or $a > c > b$

32. (a, b, c) : Given that $a \neq b \neq c$ and $a, b, c \in R$
Now, $ax^2 + bx + c \geq 0 \Rightarrow b^2 - 4ac \leq 0$ and $a > 0$ ... (i)
$bx^2 + cx + a \geq 0 \Rightarrow c^2 - 4ab \leq 0$ and $b > 0$ ... (ii)
$cx^2 + ax + b \geq 0 \Rightarrow a^2 - 4bc \leq 0$ and $c > 0$ ... (iii)
Equality cannot hold simultaneously $\because a \neq b \neq c$

$$\therefore \frac{a^2 + b^2 + c^2}{ab + bc + ca} < 4 \quad \because a \neq b \neq c$$

(from (i), (ii) and (iii))

$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2>0$

$\Rightarrow \dfrac{a^2+b^2+c^2}{ab+bc+ca}>1 \ \therefore \ \dfrac{a^2+b^2+c^2}{ab+bc+ca}\in(1,4)$

33. (a, b, c) : Sum of roots = product of roots

$\Rightarrow -\dfrac{b}{a}=\dfrac{c}{a} \Rightarrow b+c=0$

$\therefore \ \sec^2\theta+\operatorname{cosec}^2\theta=\sec^2\theta\cdot\operatorname{cosec}^2\theta$

Roots are real, $\Delta\geq 0$

$b^2-4ac\geq 0$ but $b=-c$, product of the roots >0

$\Rightarrow c^2-4ac\geq 0 \Rightarrow c(c-4a)\geq 0 \quad \because c>0$

$\Rightarrow c-4a\geq 0$

$\Rightarrow c\geq 4a \Rightarrow -b\geq 4a \Rightarrow 4a+b\leq 0$

34. (a, c) : $x^2+bx-a=0$ ...(i)

$x^2-ax+b=0$ ...(ii)

$x(a+b)-(a+b)=0$ (from (i) and (ii))

$\Rightarrow (a+b)(x-1)=0$

$\Rightarrow a+b=0, \ x=1$

By substituting $x=1$ in (i), we get $a-b=1$

35. (b, c) : Since, a, b, c, are in G.P

$\Rightarrow b^2=ac$

$\Rightarrow ax^2+\sqrt{ac}x+c=0$

$\Rightarrow x=\dfrac{-\sqrt{ac}\pm\sqrt{(\sqrt{ac})^2-4ac}}{2(a)}$

$x=\dfrac{-\sqrt{ac}\pm\sqrt{-3ac}}{2a} \ ; \ x=\dfrac{-\sqrt{ac}\pm i\sqrt{3ac}}{2a}$

$\therefore$ roots are imaginary

$\dfrac{\alpha}{\beta}=\dfrac{-\sqrt{ac}+i\sqrt{3ac}}{-\sqrt{ac}-i\sqrt{3ac}}=\dfrac{(i\sqrt{3ac}-\sqrt{ac})^2}{ac+3ac}$

$=\dfrac{-3ac+ac-2i\sqrt{3ac}}{4ac}=\dfrac{-2-2i\sqrt{3}}{4}$

$=\dfrac{-1-i\sqrt{3}}{2}=\omega^2=\dfrac{1}{\omega}$

36. (a, b, c) : $\Delta=0 \Rightarrow a(a^3+b^3+c^3-3abc)=0$

$\Rightarrow a=0$ or $a+b+c=0$

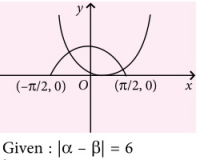
But not $a^2+b^2+c^2$ because a, b, c are all distinct.

37. (b, c, d) : $x^2-2x+\sin^2\alpha=0$

$\Rightarrow x^2-2x+1=1-\sin^2\alpha$

$\Rightarrow (x-1)^2=\cos^2\alpha$

$\Rightarrow y=(x-1)^2 \ ; \ y=\cos^2\alpha$



38. (c, d) : Given : $|\alpha-\beta|=6$

$\Rightarrow (\alpha+\beta)^2-4\alpha\beta=36$

$\Rightarrow h^2-4(7)=36 \Rightarrow h^2=36+28$

$\Rightarrow h^2=64 \Rightarrow h=\pm 8$

39. (c, d) : $(13x-1)^2=36 \Rightarrow 13x-1=\pm 6$

$\Rightarrow x=\dfrac{7}{13}$ or $x=-\dfrac{5}{13}$

But $x\neq\dfrac{7}{13}$ as $-1<x<0$

$\therefore \ x=-\dfrac{5}{13}$

$\Rightarrow \cos\alpha=-\dfrac{5}{13}; \sin\alpha=\pm\dfrac{12}{13}$

$\Rightarrow \sin 2\alpha=2\sin\alpha\cos\alpha=\pm\dfrac{120}{169}$

40. (a, b, c) : $a+b-2c=(a-c)+(b-c)>0$

$\Rightarrow$ mouth opens upwards

Now $x=1$ is a root.

Hence both roots are rational.

Vertex $=-\dfrac{(b+c-2a)}{2(a+b-2c)}>0 \Rightarrow$ vertex >0

41. (c, d) : Since α and β are the roots of $ax^2+bx+c=0$, we have

$\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

Also, since α' and β' are roots of the equation $a^3x^2+(abc)x+c^3=0$ then

$\alpha'+\beta'=\dfrac{-abc}{a^3}=\left(\dfrac{-b}{a}\right)\left(\dfrac{c}{a}\right)=(\alpha+\beta)\alpha\beta$ .....(i)

$\alpha'\beta'=\dfrac{c^3}{a^3}=(\alpha\beta)^3$ .....(ii)

Now, $(\alpha'-\beta')^2=(\alpha'+\beta')^2-4\alpha'\beta'$

$=(\alpha+\beta)^2\alpha^2\beta^2-4\alpha^3\beta^3$ (from (i) and (ii))

$=(\alpha\beta)^2[(\alpha+\beta)^2-4\alpha\beta]=(\alpha\beta)^2(\alpha-\beta)^2$

Also, $|\alpha'-\beta'|=|\alpha\beta(\alpha-\beta)|$

Therefore, $\alpha'\beta'=\alpha\beta(\alpha-\beta)$

$\Rightarrow \alpha' = \alpha^2\beta$ and $\beta' = \alpha\beta^2$
Also, $\alpha' - \beta' = -\alpha\beta(\alpha - \beta)$
$\Rightarrow \alpha' = \alpha\beta^2$ and $\beta' = \alpha^2\beta$

42. (c, d) : Let α be a root of equation $x^2 - 3x + c = 0$
Then $-\alpha$ is a root of equation $x^2 + 3x - c = 0$
$\Rightarrow c = 0 \Rightarrow$ the solutions of $x^2 - 3x = 0$ are 0 and 3.

43. (a, c) : Given equation is $(ay - bx)^2 + 4xy = 0$
$\Rightarrow a^2y^2 + b^2x^2 + (4 - 2ab)xy = 0$

$$\Rightarrow \frac{a^2y}{x} + \frac{b^2x}{y} - (2ab - 4) = 0 \qquad \text{... (i)}$$

Put $\frac{y}{x} = t$ in (i), we get
$\Rightarrow a^2t^2 - (2ab - 4)t + b^2 = 0$
$\Rightarrow D = (2ab - 4)^2 - 4a^2b^2$
$= 4(4 - 4ab) = 16(1 - ab)$
must be a perfect square for rational roots.

44. (a, b, c, d) : Let $k = \alpha + i\beta$ and one real root γ, then $(\alpha + i\beta)\gamma^2 + \gamma + 1 = 0$
Equating real and imaginary parts, $\beta\gamma^2 = 0$
$\Rightarrow \gamma = 0$ (since $\beta \neq 0$)
when $\gamma = 0 \Rightarrow 1 = 0$, which is impossible
so $k \in \phi$

45. (a, b, c, d) : a, b, c are roots of equation

$$\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2} = 1$$

$\Rightarrow (x - 1)(x - 2) + x(x - 2) + x(x - 1)$
$= x(x - 1)(x - 2)$
$\Rightarrow x^3 - 6x^2 + 8x - 2 = 0$
$\Rightarrow$ Sum of roots $= a + b + c = 6$
Product of roots $= abc = 2$

46. (b, d) : Assuming quadratic in a
$a = x^2 + 2x$, $a = x^2 - 2x + 3$
$\therefore$ given exp is $(a - x^2 - 2x)(a - x^2 + 2x - 3) = 0$
$\therefore x = -1 \pm \sqrt{1 + a}$ & $x = 1 \pm \sqrt{a - 2}$
$\Rightarrow a \geq -1$, $a \geq 2$

47. (a, c) : Suppose that both r and $r + 1$ are solutions to the equation $x^3 + 3x^2 - 34x = m$.
Then $r^3 + 3r^2 - 34r = m$, and
$(r + 1)^3 + 3(r + 1)^2 - 34(r + 1) = m$.
Subtracting the first of these equations from the second yields $(3r^2 + 3r + 1) + 3(2r + 1) - 34 = 0$, and simplification yields $3r^2 + 9r - 30 = 0$.
Thus $0 = r^2 + 3r - 10 = (r + 5)(r - 2)$
We conclude that either $r = -5$ or $r = 2$.
If $r = -5$, then $m = (-5)^3 + 3(-5)^2 - 34(-5)$
$= -125 + 75 + 170 = 120$
The other possibility is $r = 2$, which yields $m = 8 + 12 - 68 = -48$. $\therefore m = 120$ and $m = -48$ are the two possibilities.

48. (c, d) : disc ≥ 0, $a < 3$ and $f(3) > 0$
$f(x) = x^2 - 2ax + a^2 - 3$
$\Rightarrow 9 - 6a + a^2 + a - 3 > 0$ where
$\Rightarrow a^2 - 5a + 6 > 0$
$\Rightarrow (a - 3)(a - 2) > 0$
$\Rightarrow a \in (-\infty, 2) \cup (3, \infty)$

49. (b, d) : $\Delta > 0 \Rightarrow (a - b)^2 + 4(a + b) - 1 > 0$
$\Rightarrow b^2 + (4 - 2a)b + (a^2 + 4a - 4) > 0 \ \forall\, b \in R$
$\Rightarrow (4 - 2a)^2 - 4(a^2 + 4a - 4) < 0$
$\Rightarrow 32 - 32a < 0 \Rightarrow a > 1 \Rightarrow a \in (1, \infty)$

50. (a, b, d) : $-3 < \frac{x^2 + kx + 1}{x^2 + x + 1} < 3$
$\Rightarrow -3(x^2 + x + 1) < x^2 + kx + 1 < 3(x^2 + x + 1)$
$\Rightarrow 4x^2 + (k + 3)x + 4 > 0$ and
$2x^2 + (3 - k)x + 2 > 0 \ \forall\, x \in R$
$\Rightarrow (k + 3)^2 - 64 < 0$ and $(k - 3)^2 - 16 < 0$
$\Rightarrow k^2 + 6k - 55 < 0$ and $k^2 - 6k - 7 < 0$
$\Rightarrow k \in (-11, 5)$ and $k \in (-1, 7) \Rightarrow k \in (-1, 5)$
The number of integral values of R is 5
If $f(x) = x^2 - (k^2 - 4k - 5)x - 1$
Then $f(0) \times f(1) = -1(1 - (k^2 - 4r - 5) - 1)$
$= k^2 - 4k - 5 = (k + 1)(k - 5) < 0$
$\therefore$ $f(x) = 0$ has exactly one real root in $(0, 1)$

51. (a) : Difference of the roots of $f(x)$ and $g(x)$ are equal

$$\Rightarrow \frac{\Delta_1}{1} = \frac{\Delta_2}{1}$$

The least value of

$$f(x) = \frac{-\Delta_1}{4} = \frac{-1}{4} \Rightarrow \Delta_1 = 1 \Rightarrow \Delta_2 = 1$$

$\Rightarrow$ least value of $g(x)$ is also 1/4.

52. (d) : The least value of $g(x)$ occurs at $x = 7/2$

$$i.e. \frac{-b_2}{2} = \frac{7}{2} \Rightarrow b_2 = -7 \Rightarrow b_2^2 - 4c_2 = \Delta_2 \Rightarrow c_2 = 12$$

53. (c) : Roots of $x^2 - 7x + 12 = 0$ are $x = 3, 4$

54. (a) : $f(x) = x^2 - mx + 1$ and $\alpha < \beta$ are the roots of $f(x) = 0$. Now, $\alpha < \beta < 1$ implies that $f(1)$ and the coefficients of x^2 have the same sign. This gives $1 - m + 1 = f(1) > 0$

$\Rightarrow\ m < 2$ ....(1)

Also, discriminant is $m^2 - 4 \geq 0$. Therefore

$m \leq -2$ or $m \geq 2$ ....(2)

From Eqs. (1) and (2), $m \leq -2$.

Also, note that if $m = -2$, the roots are $-1, -1$.

55. (d) : Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$.

Let $\beta > \alpha > 3$ be the roots of $f(x) = 0$.

Then $6 < \alpha + \beta = 6m$ and hence $m > 1$.

Also, $9m^2 - 2m + 2 = \alpha\beta > 9$

Therefore, $9m^2 - 2m - 7 > 0$

$\Rightarrow\ (9m + 7)(m - 1) > 0$

This gives, $m < \frac{-7}{9}$ or $m > 1$

Also, $f(3)$ and the coefficient of x^2 have the same sign. Therefore, $f(3) > 0$. This gives

$9 - 18m + 9m^2 - 2m + 2 > 0$

$\Rightarrow\ 9m^2 - 20m + 11 > 0$

$\Rightarrow\ (9m - 11)(m - 1) > 0$

$\Rightarrow\ m < 1$ or $m > \frac{11}{9}$, we get $\frac{11}{9} < m$

56. (d) : Let α, β, where $\alpha \leq \beta$, be the roots of $4x^2 - 2x + m = 0$.

Then $-1 < \alpha, \beta < 1$ and $\alpha + \beta = \frac{1}{2}, \alpha\beta = \frac{m}{4}$

Now, $f(-1)$ and the coefficient of x^2 have the same sign. Therefore, $f(-1) > 0$ and hence $4 + 2 + m > 0$.

That is $m > -6$ ... (1)

Also, $f(1) > 0 \Rightarrow 4 - 2 + m > 0$.

This implies $m > -2$ ... (2)

The discriminant is $4 - 16m \geq 0$.

Therefore $m \leq 1/4$ ... (3)

From Eqs. (1) – (3), we get $-2 < m \leq \frac{1}{4}$

If $m = 1/4$, then the given equation is

$$4x^2 - 2x + \frac{1}{4} = 0 \text{ or } 16x^2 - 8x + 1 = 0$$

Therefore, the roots are $1/4, 1/4 \in (-1, 1)$.

If the roots are distinct, then $< -2 < m < 1/4$.

57. (b) : $x^2 + 19x + 92 =$ a perfect square

$\Rightarrow\ 4x^2 + 76x + 368 = 4\ m^2$

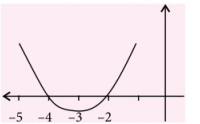
$\Rightarrow\ (2x + 19)^2 = m^2 - 7$

$\Rightarrow\ -(2x + 19)^2 + m^2 = 7 \times 1$, 7 is a prime number.

$m_1 = -2,\ m_2 = 2$

58. (c)

59. (a) : Since roots of $x^2 + 19x + 92 - m^2 = 0$ are integers $\therefore\ D = 19 \cdot 19 - 4(92 - m^2)$ must be a perfect square.



$\Rightarrow\ 4m^2 - 7$ must be a perfect square

$\Rightarrow\ m_1 = -2$ and $m_2 = 2$

Clearly, $m_1^2 + m_2^2 = 8$

Now, the given quadratic expression is

$f(x) = (x + 2)(x + a^2 + 1)$

Clearly, one root is -2.

Other root is $-5 \leq (a^2 + 1) < -4$.

Hence, $f(x)$ will be negative for exactly two integral values of x, hence graph is shown above.

60. (c) : Let $abc + d = \frac{abcd}{d} + d$

$abc + d$ and all its cyclic counterparts are equal to 2. Put $\lambda = abcd$ and consider the equation

$$\frac{\lambda}{x} + x = 2 \quad \text{... (1)}$$

This is satisfied by $x = a, b, c, d$.

Writing equation (1) as

$x^2 - 2x + \lambda = 0$, we have $P = 1$.

61. (b) : If a, b, c, d are not all equal, then there are essentially two cases, *viz*

Case 1: $a = b = c$ and d is different from them.

Case 2 : $a = b$ and $c = d$, but a and c are distinct.

In case 1, $a^3 + d = 2$ and $a^2 d + a = 2$

Eliminate d and solve to get $a = -1$, $d = 3$

Show that case 2 yields no solution.

62. (c) : The solutions are

$(a, b, c, d) = (1, 1, 1, 1), (-1, -1, -1, 3)$

and all the permutations of these.

63. (c) **64. (a)** **65. (c)**

Let $f(x) = l_0(x) + l_1(x) + l_2(x)$

Put $x = x_0 \Rightarrow f(x_0) = 1$

$x = x_1 \Rightarrow f(x_1) = 1$

$x = x_2 \Rightarrow f(x_2) = 1$

$\therefore\ f(x) = 1$ for $x = x_0, x_1, x_2$

$$\sum_{k=0}^{2} l_k(x)\cdot x^2 = x^2$$

Given,

$P(x) = (x_1 + x_2)l_0(x) + (x_2 + x_0)l_1(x) + (x_0 + x_1)l_2(x)$

Put $x = x_0$, $P(x_0) = (x_1 + x_2)\cdot 1 + 0 + 0$

$= x_0 + x_1 + x_2 - x_0$

$x = x_1$, $P(x_1) = x_2 + x_0 = x_0 + x_1 + x_2 - x_1$

$x = x_2$, $P(x_2) = x_0 + x_1 = x_0 + x_1 + x_2 - x_2$

Hence, $P(x) = x_0 + x_1 + x_2 - x$

Given, $l(x) = (x - x_0)(x - x_1)(x - x_2)$

$$\frac{l(x)}{(x-x_0)l'(x_0)} = l_0(x)$$

66. (d) **67. (c)** **68. (b)**

Co-ordinate of A is $(-2\sqrt{2}, 0)$

Co-ordinate of C is $(2\sqrt{2}, 0)$

$AB = BC = 4$ units, $OB = 2\sqrt{2}$

$$y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$$

Minimum value of $y = f(x)$ is $-2\sqrt{2}$ at $x = 0$ for $f(x) = 0$

$$\frac{x^2}{2\sqrt{2}} - 2\sqrt{2} = 0$$

$\Rightarrow x^2 - 8 = 0 \Rightarrow x = \pm 2\sqrt{2}$

Number of integral values of 'K' which lies in $(-2\sqrt{2}, 2\sqrt{2})$ is 5.

69. A-q; B-p; C-s; D-r

(A) $ax^2 - ax + 1 > 0 \ \forall \ x \in R$

$\Rightarrow D < 0 \Rightarrow a^2 - 4a < 0 \Rightarrow a \in (0, 4)$

(B) Let $f(x) = x^3 - 3x + \frac{a}{2}$

$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$

Now, $f(1)\, f(-1) < 0$

$$\Rightarrow \left(\frac{a}{2} - 2\right)\left(\frac{a}{2} + 2\right) < 0$$

$\Rightarrow -4 < a < 4$

(C) Let $f(x) = x^3 + ax^2 + x + 1$

$f'(x) = 3x^2 + 2ax + 1$

$f'(x) \geq 0 \Rightarrow D \leq 0 \Rightarrow 4a^2 - 12 \leq 0$

$\Rightarrow a^2 \leq 3$

(D) Product of roots $= a^2 - 9 < 0$

$\Rightarrow -3 < a < 3$

70. A-p, q, r ; B-p, q, r ; C-p, q; D-p, q, s

(A) Consider the polynomial

$P(x) = ax^3 + bx^2 + cx$

and use Rolle's theorem on $[0, 1]$.

(B) Consider the polynomial

$$P(x) = \frac{1}{3}ax^3 + \frac{1}{2}bx^2 + \frac{1}{6}cx$$

and use Rolle's theorem on $[0, 1]$.

(C) Let $P(x) = ax^2 + bx + c$

Note that, $P(-1)P(1) < 0$

$\Rightarrow$ $P(x)$ vanishes at least once in $(-1, 1)$

$\Rightarrow$ $P(x) = 0$ has real roots.

(D) Let $P(x) = ax^2 + bx + c$ and

$$Q(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$$

Since, $1 + \frac{|b|}{a} + \frac{c}{a} < 0$, we get $Q(-1)$ or $Q(1) < 0$

Since $Q(x) \to \infty$ as $x \to \pm\infty$, we get $Q(x) = 0$ or $P(x) = 0$ has a root in $(-\infty, -1)$ or $(1, \infty)$.

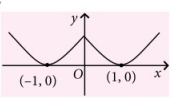
71. A - s; B - q; C - r; D - p

Given equation, $|x|^2 + a|x| + 1 = 0$

(A) If $a \geq 0$, vertex $= \left(\frac{-a}{2}, \frac{4 - a^2}{4}\right)$

Expression is positive $\therefore$ No solution

(B) If $a = -2$

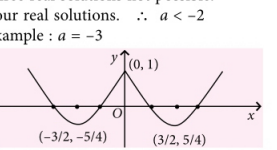


$\therefore$ Two real solutions

(C) Three real solutions not possible.

(D) Four real solutions. $\therefore$ $a < -2$

For example : $a = -3$



72. A-p, s, t; B-p, r, s, t; C-q; D-p

(A) Let $f(x) = x^2 - 6kx + 2 - 2k + 9k^2$ be such that α and β be the roots of $f(x) = 0$. Then, since $3 < \alpha$ and $3 < \beta$, we have, $6k > 6$ and therefore $k > 1$.

Also, $f(3) > 0$

$\Rightarrow 9k^2 - 20k + 1 > 0$ or $(9k - 11)(k - 1) > 0$

or $k < 1$ or $k > 11/9$

Since, $k > 1$, it follows that $k > 11/9$.

(B) The inequality is defined for $x > 1$. Since $\log_2(x-1) = \log_{0.5^{-1}}(x-1) = -\log_{0.5}(x-1)$ ...(i) we have,

$$\log_{0.1}(x^2+x) > \log_{0.5}\left(\frac{x^3-x}{x-1}\right) = \log_{0.5}(x^2+x)$$

(from (i))

which further gives $x^2 + x > 1 \Rightarrow x^2 + x - 1 > 0$

$$\Rightarrow x < -\left(\frac{1+\sqrt{5}}{2}\right) \text{ or } x > \frac{\sqrt{5}-1}{2} \text{ and } x > 1$$

This implies $x \in (1, \infty)$

(C) We have, $|x - 1 - x^2| \le x^2 - 3x + 4|$

$\Rightarrow |x^2 - x + 1| \le |x^2 - 3x + 4|$

$\Rightarrow x^2 - x + 1 \le x^2 - 3x + 4$

(since both are positive for all real x)

$$\Rightarrow x \le \frac{3}{2}. \quad \text{Therefore, } x \in \left(-\infty, \frac{3}{2}\right)$$

(D) We have, $|x^2 - 2x - 3| < 3x - 3$

$\Rightarrow |(x-3)(x+1)| < 3x - 3$

Case 1 : $x < -1$, then

$(x-3)(x+1) < 3x - 3 \Leftrightarrow x^2 - 5x < 0$

$\Leftrightarrow 0 < x < 5$. However, $x < -1$

Case 2 : $-1 < x < 3$. Then

$(3-x)(x+1) < 3x - 3$

$\Leftrightarrow -x^2 + 2x + 3 < 3x - 3 \Leftrightarrow 0 < x^2 + x - 6$

$\Leftrightarrow (x+3)(x-2) > 0$

Therefore, $x > 2$, hence $x \in (2, +\infty)$

Case 3 : $x \ge 3$. Then

$(x-3)(x+1) < 3x - 3 \Leftrightarrow x^2 - 2x - 3 < 3x - 3$

$\Leftrightarrow x^2 - 5x < 0 \Rightarrow 0 < x < 5$

$\Rightarrow 3 \le x < 5$ ($\because x \ge 3$)

From the above two cases, $x \in (2, 5)$

73. A - s ; B - r ; C - p ; D - p

(A) Let the roots be r, s, and t. Then they satisfy $r + s + t = -a$, $rs + st + rt = b$, and $rst = -c$.

So we have

$-(a + b + c + 1) = r + s + t - rs - rt - st + rst - 1$

$= (r-1)(s-1)(t-1) = 2009 = 7 * 7 * 41$.

Thus, the roots are 8, 8, and 42, and

$a = -(r + s + t) = -58$.

(B) The polynomial $f(x) = P(x) - x - 2006$ satisfies $f(0) = f(1) = f(2) = 0$, so it has $x(x-1)(x-2)$ as a factor. It has degree 4, so there is one more factor, call it $x - l$.

Then, $P(x) = x + 2006 + x(x-1)(x-2)(x-l)$.

The coefficient of x^3 is $-3 - l$, so l is an integer. We have, $P(3) = 2009 + 6(3 - l) = 2027 - 6l$. The smallest positive value for this expression is when $l = 337$, which yields the value 5.

(C) $\dfrac{\alpha}{\beta} + \dfrac{\bar{\alpha}}{\bar{\beta}} = 2 \Rightarrow \text{Re}\left(\dfrac{\alpha}{\beta}\right) = 1$

$$\Rightarrow \frac{\alpha}{\beta} - 1 = ki \ (k \ne 0 \in R)$$

$$\Rightarrow \frac{\alpha - \beta}{\beta} \in ki$$

$$\Rightarrow \angle OBA = \text{Arg}\left(\frac{\alpha-\beta}{\beta}\right) = \frac{\pi}{2}$$

(D) $r_1 + r_2 + ... + r_n = 56$ and

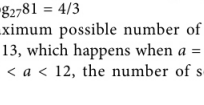
$r_1 r_2 ... r_n = 2009 = 7^2 \times 41$

So, none of r_i can be 49 or bigger factor of 2009, otherwise their sum is bigger than 56. One r_i is 41, two r_i's are 7, and the remaining r_i's are 1.

Since $56 = 41 + 7 + 7 + 1$, we have $n = 4$.

74. A - r ; B - r ; C - p ; D - s

The line $y = k$, $k < 12$ doesn't meet the above curve



$k = 2$ meets the curve at infinite points belonging to $[-9, -3]$

$k > 12$ meets the curve at two points

For $k = 12$, the line $y = k$ meets the curve $y = |x - 3| + |x + 9|$ meets at 13 integral points.

Note that $\log_{27}81 = 4/3$

(A) The maximum possible number of integral solutions is 13, which happens when $a = 12$

(B) For $4/3 < a < 12$, the number of solutions = 0,

for $a = 12$, the number of solutions is infinite,

for $a > 12$, the number of solutions is two.

75. (2) : $x = -4, -1 - \sqrt{3}$ are solutions

76. (0) : Since cubic polynomial is divisible by both $x^2 + ax + b$ and $x^2 + bx + a$.

Therefore, $x^2 + ax + b$ and $x^2 + bx + a$ must have a common root

$x^2 + ax + b = 0$

$x^2 + bx + a = 0$

Subtract

$\Rightarrow x(a - b) = (a - b) \Rightarrow x = 1$

Therefore, common roots

for $x^2 + ax + b = 0$ are 1 and α

$\Rightarrow 1 \cdot \alpha = b \Rightarrow \alpha = b$

for $x^2 + bx + a = 0$ are 1 and β

$\Rightarrow 1 \cdot \beta = a \Rightarrow \beta = a$

$\Rightarrow$ roots of cubic be 1, a, b

$\Rightarrow 1 \times a \times b = -72$ ... (1)

and $a + b + 1 = 0$ ... (2)

(from $x^2 + ax + b = 0$ put $x = 1$)

$\Rightarrow a - \frac{72}{a} = -1 \Rightarrow a^2 + a - 72 = 0$

$\Rightarrow (a + 9)(a - 8) = 0 \Rightarrow a = -9, 8$

Therefore roots are 1, −9, 8

77. (1) : $\Delta \geq 0, -\frac{b}{2a} < 3$ and $f(3) > 0$

$\Rightarrow a \leq 3$, $a < 3$ and $a \in (-\infty, 2) \cup (3, \infty)$

$\Rightarrow$ The maximum value of $[a]$ is 1

78. (4) : $t^2 - 9t + 14 = 0 \Rightarrow (t - 2)(t - 7) = 0$

$\Rightarrow [x^2 - 2] = 2$ or $[x^2 - 2] = 7$

$\Rightarrow 2 \leq x^2 - 2 < 3$ or $7 \leq x^2 - 2 < 8$

$\Rightarrow 4 \leq x^2 < 5$ or $9 \leq x^2 < 10$

$\Rightarrow 2 \leq |x| < \sqrt{5}$ or $3 \leq |x| < \sqrt{10}$

$\Rightarrow x \in (-\sqrt{5}, -2] \cup [2, \sqrt{5})$

or $x \in (-\sqrt{10}, -3] \cup [3, \sqrt{10})$

$\Rightarrow$ The integral values of x are −2, 2, −3, 3

79. (1) : $(a + c)^2 - b^2 < 0$

$\Rightarrow (a - b + c)(a + b + c) < 0$

$\Rightarrow f(-1) f(1) < 0$

$\Rightarrow f(x) = 0$ has exactly one solution in $(-1, 1)$

80. (1) : $(x - 1)(x - a) = 1$

$x - 1 = 1$, $x - a = 1$ or $x - 1 = -1$, $x - a = -1$

$\Rightarrow x = 2$, $a = 1$ or $x = 0$, $a = 1$

81. (8) : Discriminant of $x^2 + 3x + 5 = 0$ is negative.

Therefore the roots of $x^2 + 3x + 5 = 0$ are complex conjugate pairs. Since the equations have a common root, the common root is complex and hence both the (complex) roots are common. Therefore the coefficients of the equations are proportional.

$\therefore \frac{a}{1} = \frac{b}{3} = \frac{2}{5}$

$\therefore a = \frac{2}{5}$ and $b = \frac{6}{5}$ $\therefore a + b = \frac{8}{5}$

82. (1) : Let $f(x) = x^2 + 2(m - 1)x + m + 5$

Case (i) : $f(x) > 0 \ \forall x$ if $\Delta < 0 \Rightarrow m \in (-1, 4)$ ... (1)

Case (ii) : $f(x) > 0 \ \forall x$ if both roots are smaller than 1

$\Delta > 0$, $f(1) > 0$, $\frac{-b}{2a} < 1$

$\therefore m \in (4, \infty)$ ... (2)

Taking union of (1) and (2)

$m \in (-1, \infty)$. $\therefore k = 1$

83. (1) : $f(x)$ = constant function

$\therefore f(x) = 3$

Given expression is $3(x^2 + 2x + 1) = 0$

$\therefore x = -1$ is the only one solution.

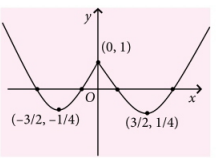
84. (6) : Subtracting the two equations, we get the common root as $x = -\frac{1}{2}(\alpha + \beta)$. Substituting this in any equation, we get $(\alpha - \beta)^2 = 4$ *i.e.* $|\alpha - \beta| = 2$

Now, $2\alpha^2 - 4\alpha\beta - |\alpha - \beta| + |\alpha - \beta|\beta^2$

$= 2\alpha^2 - 4\alpha\beta + 2\beta^2 - 2 = 2(\alpha - \beta)^2 - 2$

$= 2.4 - 2 = 8 - 2 = 6$

85. (4) :



Alternative solution :

$|x|^2 - 3|x| + 2 = |x - 1| \ |x - 2|$

$\Rightarrow |x| = 1, 2$

So, there are four solutions.

86. (7) : For the equation $px^2 + qx + 1 = 0$, to have real roots $D \geq 0 \Rightarrow q^2 \geq 4p$

If $p = 1$ then $q^2 \geq 4 \Rightarrow q = 2, 3, 4$

If $p = 2$ then $q^2 \geq 8 \Rightarrow q = 3, 4$

If $p = 3$ then $q^2 \geq 12 \Rightarrow q = 4$

If $p = 4$ then $q^2 \geq 16 \Rightarrow q = 4$

$\therefore$ Number of required equations = 7.

concept BOOSTERS

Class XII

Functions

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This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DEFINITION

If to every value (considered as real unless otherwise stated) of a variable x, which belongs to some collection (set) E, there corresponds one and only one finite value of the quantity y, then y is said to be a function (single valued) of x or a dependent variable defined on the set E; x is the argument or independent variable.

If to every value of x belonging to some set E there corresponds one or several values of the variable y, then y is called a multiple valued function of x defined on E. Conventionally the word "FUNCTION" is used only as the meaning of a single valued function, if not otherwise stated.

Pictorially : $\xrightarrow[\text{input}]{x} \boxed{f} \xrightarrow[\text{input}]{f(x)=y}$, y is called the image of x & x is the pre-image of y under f.

Every function from $A \to B$ satisfies the following conditions :

- $\forall\ a \in A \Rightarrow (a, f(a)) \in f$
- $f \subset A \times B$ and
- $(a, b) \in f\ \ \&\ \ (a, c) \in f \Rightarrow b = c$

DOMAIN, CO-DOMAIN AND RANGE OF A FUNCTION

Let $f : A \to B$, then the set A is known as the domain of f & the set B is known as co-domain of f. The set of all f images of elements of A is known as the range of f. Thus :

Domain of $f = \{a \mid a \in A(a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$

It should be noted that range is a subset of co-domain. If only the rule of function is given then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range.

IMPORTANT TYPES OF FUNCTIONS

- **Polynomial Function :**

  If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

  (a) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax$, $a \neq 0$.

  (b) There are two polynomial functions, satisfying the relation $f(x).f(1/x) = f(x) + f(1/x)$. They are :
  (i) $f(x) = x^n + 1$ (ii) $f(x) = 1 - x^n$,
  where n is a positive integer .

- **Algebraic Function :**

  y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x)y^n + P_1(x)y^{n-1} + \ldots + P_{n-1}(x)y + P_n(x) = 0$. Where n is a positive integer and $P_0(x)$, $P_1(x)$, ... are polynomials in x.

  e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

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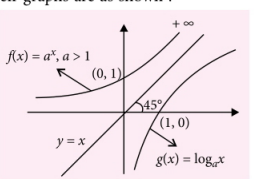
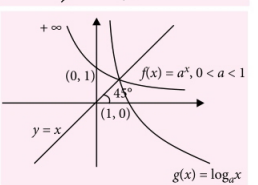
Note that all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called Transcedental Function.

- **Fractional Rational Function :**
A rational function is a function of the form $y = f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ & $h(x)$ are polynomials & $h(x) \neq 0$.

- **Exponential Function :**
A function $f(x) = a^x = e^{x \ln a}$ ($a > 0$, $a \neq 1$, $x \in R$) is called an exponential function. The inverse of the exponential function is called the logarithmic function. i.e. $g(x) = \log_a x$.
Note that $f(x)$ & $g(x)$ are inverse of each other & their graphs are as shown :





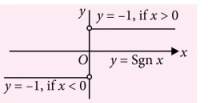
- **Absolute Value Function :**
A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as :

$$y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

- **Signum Function :**
A function $y = f(x) = \text{Sgn}(x)$ is defined as follows :

$$y = f(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$

It is also written as $\text{Sgn } x = |x|/x$; $x \neq 0$; $f(0) = 0$



- **Greatest Integer Or Step Up Function :**
The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x. Note that for :

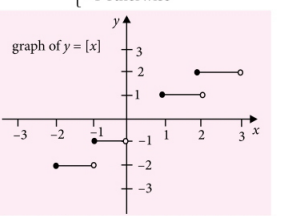
$-1 \leq x < 0$; $[x] = -1$, $0 \leq x < 1$; $[x] = 0$

$1 \leq x < 2$; $[x] = 1$, $2 \leq x < 3$; $[x] = 2$

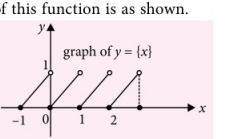
and so on.

Properties of greatest integer function :

- $[x] \leq x < [x] + 1$ and
- $x - 1 < [x] \leq x$, $0 \leq x - [x] < 1$
- $[x + m] = [x] + m$, if m is an integer.
- $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$
- $[x] + [-x] = \begin{cases} 0 \text{ if } x \text{ is an integer} \\ -1 \text{ otherwise} \end{cases}$



- **Fractional Part Function :**
It is defined as :
$g(x) = \{x\} = x - [x]$.
e.g. the fractional part of the number 2.1 is $2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3. The period of this function is 1 and graph of this function is as shown.



DOMAINS AND RANGES OF COMMON FUNCTION

Function ($y = f(x)$)	Domain (i.e. values taken by x)	Range (i.e. values taken by $f(x)$)
x^n, $(n \in N)$	R = (set of real numbers)	R, if n is odd $R^+ \cup \{0\}$, if n is even
$\frac{1}{x^n}, (n \in N)$	$R - \{0\}$	$R - \{0\}$, if n is odd R^+, if n is even
$x^{1/n}$, $(n \in N)$	R, if n is odd $R^+ \cup \{0\}$, if n is even	R, if n is odd $R^+ \cup \{0\}$, if n is even
$\frac{1}{x^{1/n}}, (n \in N)$	$R - \{0\}$, if n is odd R^+, if n is even	$R - \{0\}$, if n is odd R^+, if n is even
$\sin x$	R	$[-1, +1]$
$\cos x$	R	$[-1, +1]$
$\tan x$	$R - (2k+1)\frac{\pi}{2}, k \in I$	R
$\sec x$	$R - (2k+1)\frac{\pi}{2}, k \in I$	$(-\infty, -1] \cup [1, \infty)$
$\operatorname{cosec} x$	$R - k\pi$, $k \in I$	$(-\infty, -1] \cup [1, \infty)$
$\cot x$	$R - k\pi$, $k \in I$	R
$\sin^{-1} x$	$[-1, +1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, +1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
$\cot^{-1} x$	R	$(0, \pi)$
e^x	R	R^+
$e^{1/x}$	$R - \{0\}$	$R^+ - \{1\}$
a^x, $a > 0$	R	R^+
$a^{1/x}$, $a > 0$	$R - \{0\}$	$R^+ - \{1\}$
$\log_a x$, $(a > 0)$ $(a \neq 1)$	R^+	R

$\log_x a = \frac{1}{\log_a x}$ $(a > 0)(a \neq 1)$	$R^+ - \{1\}$	$R - \{0\}$
$[x]$	R	I
$\frac{1}{[x]}$	$R - [0, 1)$	$\left\{\frac{1}{n}, n \in I - \{0\}\right\}$
$\{x\}$	R	$[0, 1)$
$\frac{1}{\{x\}}$	$R - I$	$(1, \infty)$
$\lvert x\rvert$	R	$R^+ \cup \{0\}$
$\frac{1}{\lvert x\rvert}$	$R - \{0\}$	R^+
$\text{sgn}(x) = \frac{\lvert x\rvert}{x}$, $x \neq 0$	$R = 0$, $x = 0$	$\{-1, 0, 1\}$
say $f(x) = c$	R	$\{c\}$

EQUAL OR IDENTICAL FUNCTIONS

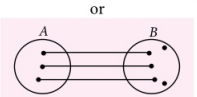
Two functions f & g are said to be equal if :

- The domain of f = the domain of g
- The range of f = the range of g
- $f(x) = g(x)$, for every x belonging to their common domain. e.g., $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$ are identical functions.

CLASSIFICATION OF FUNCTIONS

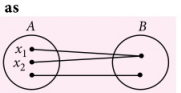
- **One-One Function (Injective mapping) :**
  A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$,
  $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.
  Diagramatically an injective mapping can be shown as

or



Remarks :

- Any function which is entirely increasing or decreasing in whole domain, then $f(x)$ is one-one.
- If any line parallel to x-axis cuts the graph of the function atmost at one point, then the function is one-one.
- **Many–one function :**
  A function $f : A \to B$ is said to be a many one function if two or more elements of A ave the same f image in B. Thus $f : A \to B$ is many one if for $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.
  Diagramatically a many one mapping can be shown as



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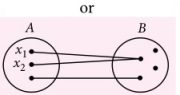


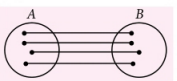
Remarks :

- Any continuous function which has atleast one local maximum or local minimum, then $f(x)$ is many-one. In other words, if a line parallel to

x-axis cuts the graph of the function atleast at two points, then f is many-one.

- If a function is one-one, it cannot be many-one and vice versa.

- **Onto function (Surjective mapping) :**
  If the function $f : A \to B$ is such that each element in B (co-domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus $f : A \to B$ is surjective iff $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.
  Diagramatically surjective mapping can be shown as



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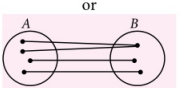
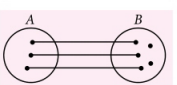


  Note : if range = co-domain, then $f(x)$ is onto.

- **Into function :**
  If $f : A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then $f(x)$ is into.
  Diagramatically into function can be shown as



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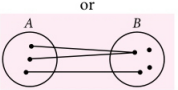
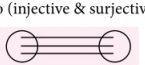


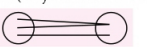
  Note : If a function is onto, it cannot be into and vice versa. A polynomial of degree even will always be into.
  Thus a function can be one of these four types :

- one-one onto (injective & surjective)



- one-one into (injective but not surjective)

- many-one onto (surjective but not injective)



- many-one into (neither surjective nor injective)

Remarks :

- If f is both injective & surjective, then it is called a Bijective mapping.
- The bijective functions are also named as invertible, non-singular functions.
- If a set A contains n distinct elements then the umber of different functions defined from $A \to A$ is n^n & out of it $n!$ are one-one.

- **Identity function :**
  The function $f : A \to A$ defined by $f(x) = x\ \forall\ x \in A$ is called the identity of A and is denoted by I_A.
  It is easy to observe that identity function is a bijection.

- **Constant function :**
  A function $f : A \to B$ is said to be a constant function if every element of A has the same f image in B. Thus $f : A \to B$; $f(x) = c$, $\forall\ x \in A$, $c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one-one or many-one, onto or into.

ALGEBRAIC OPERATIONS ON FUNCTIONS

If f & g are real valued functions of x with domain set A, B respectively, then both f & g are defined in $A \cap B$. Now we define $f + g$, $f - g$, $(f \cdot g)$ & (f/g) as follows :

- $(f \pm g)(x) = f(x) \pm g(x)$
- $(f \cdot g)(x) = f(x) \cdot g(x)$

  domain in each case is $A \cap B$

- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ domain is $\{x \mid x \in A \cap B$ such that $g(x) \neq 0\}$.

COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTIONS

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x)) \ \forall \ x \in A$ is called the composite of the two functions f & g.

Diagramatically $\xrightarrow{x}$ [f] $\xrightarrow{f(x)}$ [g] $\longrightarrow g(f(x))$.

Thus the image of every $x \in A$ under the function gof is the g-image of the f-image of x.

Note that gof is defined only if $\forall \ x \in A$, $f(x)$ is an element of the domain of g so that we can take its g-image. Hence for the product gof of two functions f & g, the range of f must be a subset of the domain of g.

Properties Of Composite Functions :

- The composite functions is not commutative i.e. $gof \neq fog$.
- The composite functions is associative i.e. if f, g, h are three functions such that $fo(goh)$ & (fog) oh are defined, then fo $(goh) = (fog)$ oh.
- The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

HOMOGENEOUS FUNCTIONS

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y. Symbolically if,

$f(tx, ty) = t^n . f(x, y)$ then $f(x, y)$ is homogeneous function of degree n.

BOUNDED FUNCTION

A function is said to be bounded if $|f(x)| \leq M$, where M is a finite quantity.

IMPLICIT AND EXPLICIT FUNCTION

A function defined by an equation not solved for the dependent variable is called an **Implicit Function**. For e.g. the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit Function**.

INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one-one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \ \forall \ x \in A$ & $y \in B$, then g is said to be inverse of f. Thus $g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Properties Of Inverse Function :

- The inverse of a bijection is unique.
- If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively.

  Note that the graphs of f & g are the mirror images of each other in the line $y = x$. As shown in the figure given below a point (x', y') corresponding to $y = x^2$ $(x \geq 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

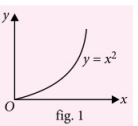


fig. 1

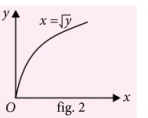


fig. 2

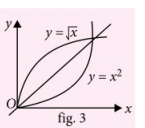


fig. 3

- The inverse of a bijection is also a bijection.
- If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} \ og^{-1}$.

ODD AND EVEN FUNCTIONS

If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.

e.g. $f(x) = \cos x$; $g(x) = x^2 + 3$.

If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.

e.g. $f(x) = \sin x$; $g(x) = x^3 + x$.

Remarks :

- The only function which is defined on the entire number line and is even and odd at the same time is $f(x) = 0$.

- Every function can be expressed as the sum of an even and an odd function.

e.g. $f(x)=\underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}}+\underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$

- Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.
- $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even and $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.
- A function may neither be odd nor even.
- Inverse of an even function is not defined.
- If f and g both are even or both are odd then the function fg will be even but if any one of them is odd then fg will be odd.

PERIODIC FUNCTION

A function $f(x)$ is called periodic if there exists a positive number T ($T > 0$) called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of x.
e.g. The function $\sin x$ & $\cos x$ both are periodic over 2π & $\tan x$ is periodic over π.

Remarks :

- If $f(x)$ has a period T & $g(x)$ also has a period T then it does not mean that $f(x) + g(x)$ must have a period T. e.g. $f(x) = |\sin x| + |\cos x|$.
- Every constant function is always periodic with no fundamental period.
- $f(T) = f(0) = f(-T)$, where 'T' is the period.
- Inverse of a periodic function does not exist.
- If $f(x)$ has a period T then $f(ax + b)$ has a period T/a ($a > 0$).
- If $f(x)$ has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ and also has a period p.

Note :

If x, y are independent variables, then :

- $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, n \in R$
- $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.
- $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.
- $f(x + y) = f(x)f(y) \Rightarrow f(x) = a^{kx}$.

SECTION-I

Single Correct Answer Type

1. The range of the function $f(x) = [\sin x + \cos x]$ (where [.] denotes the greatest integer function) is
(a) $[-2, 1]$ (b) $\{-2, -1, 0, 1\}$
(c) $\{-1, 1\}$ (d) $\{-2, -1, 1\}$

2. The range of the function $f(x)=\frac{e^x \ln x \, 5^{(x^2+2)}(x^2-7x+10)}{2x^2-11x+12}$ is
(a) $(-\infty, \infty)$ (b) $[0, \infty)$
(c) $\left(\frac{3}{2}, \infty\right)$ (d) $\left(\frac{3}{2}, 4\right)$

3. Domain of the function $f(x)=\frac{1}{\sqrt{\ln \cot^{-1} x}}$ is
(a) $(\cot 1, \infty)$ (b) $R - \{\cot 1\}$
(c) $(-\infty, 0) \cup (0, \cot 1)$ (d) $(-\infty, \cot 1)$

4. If $f(x)=\frac{x^3+x-2}{x^3-1}$ and $g(x)=\frac{x^2+x+2}{x^2+x+1}$ then (D_f represents domain and R_f represents range of $f(x)$)

(a) $D_f = R - \{1\}$ $R_f = \left(1, \frac{7}{3}\right] - \left\{\frac{4}{3}\right\}$
$D_g = R$ $R_g = \left(1, \frac{7}{3}\right]$

(b) $D_f = R - \{1\}$ $R_f = \left(1, \frac{7}{3}\right]$
$D_g = R$ $R_g = \left[1, \frac{7}{3}\right]$

(c) $D_f = R - \{1\}$ $R_f = \left[1, \frac{7}{3}\right]$
$D_g = R$ $R_g = \left[1, \frac{7}{3}\right]$

(d) $D_f = R - \{1\}$ $R_f = \left(1, \frac{7}{3}\right)$
$D_g = R$ $R_g = \left(1, \frac{7}{3}\right]$

5. If $f(x) = \sin^{-1} x$ and $g(x) = [\sin x(\cos x)] + [\cos(\sin x)]$, then range of $f(g(x))$ is

(a) $\left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$ (b) $\left\{-\frac{\pi}{2}, 0\right\}$

(c) $\left\{0, \frac{\pi}{2}\right\}$ (d) $\left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$

6. The domain of the function $\sqrt{1-f(x)}$ where $f^3(x) = 1 - x^3 - 3xf(x)\ \forall\ x \in R$ is $(f(1) \neq -1)$

(a) R (b) $R - \{0\}$

(c) $R^+ \cup \{0\}$ (d) R^-

7. Range of $f(x) = \frac{\tan(\pi[x^2 - x])}{1 + \sin(\cos x)}$ is (where $[\cdot]$ denotes the greatest integer function

(a) $(-\infty, \infty) \sim [0, \tan 1]$ (b) $(-\infty, \infty) \sim [\tan 2, 0)$

(c) $[\tan 2, \tan 1]$ (d) $(-\infty, \infty)$

8. Let $f(x) = \frac{x - [x]}{1 + x - [x]}$, then the range of f is

(a) $[0, 1]$ (b) $[0, 1/2]$

(c) $[0, 1/2)$ (d) $(0, 1)$

9. Range of $f(x) = 3 \tan \sqrt{\frac{\pi^2}{9} - x^2}$ is

(a) $[-3\sqrt{3}, 3\sqrt{3}]$ (b) $[0, \sqrt{3}]$

(c) $[0, 3\sqrt{3}]$ (d) none of these

10. If $[2 \sin x] + [\cos x] = -3$, then the range of the function $f(x) = \sin x + \sqrt{3} \cos x$ in $[0, 2\pi]$ is (where [.] denotes the greatest integer function)

(a) $[-2, 1)$ (b) $(-2, -1)$

(c) $\left(-1, -\frac{1}{2}\right)$ (d) none of these

11. The range of α such that range of $f(x) = \frac{(x^3 - 3x + 6)(x - \alpha)}{2(x - \alpha)}$ is a proper subset of R is

(a) $R - [-2, 2]$ (b) $R - (-2, 2)$

(c) $[-2, 2]$ (d) $(-2, 2)$

12. If $f(x) = \sqrt{\cos(\sin x)} + \sqrt{\sin(\cos x)}$, then the range of $f(x)$ is

(a) $[\sqrt{\cos 1}, \sqrt{\sin 1}]$ (b) $[\sqrt{\cos 1}, 1 + \sqrt{\sin 1}]$

(c) $[1 - \sqrt{\cos 1}, \sqrt{\sin 1}]$ (d) $[\sqrt{\cos 1}, 1]$

13. The range of the function $f(x) = 4\cos^3 x - 8 \cos^2 x + 1$ is

(a) $[-11, 1]$ (b) R

(c) $[-1, 1]$ (d) $[-11, -3]$

14. If $a^2 + b^2 + c^2 = 1$, then $ab + bc + ca$ lies in the interval

(a) $\left[0, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, 1\right]$ (c) $[0, 1]$ (d) $[1, 2]$

15. The domain of $\frac{\log_2(x+3)}{x^2 + 3x + 2}$ is

(a) $R - \{-1, -2\}$ (b) $[-2, \infty]$

(c) $R - \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$

16. If $f(x) = 2 \sin x + bx$ be a non-monotonic function $\forall\ x \in R$ then domain of $f(x) = \sqrt{2x^4 + bx^3 - 6x^2 - 4bx - 8}$ is

(a) $(-\infty, -2] \cup [2, \infty)$ (b) $R \sim \{-2, 2, 0\}$

(c) $R \sim [-2, 2]$ (d) $R \sim (-1, 1)$

17. If $0 \leq x \leq \frac{\pi}{3}$ then range of $f(x) = \sec\left(\frac{\pi}{6} - x\right) + \sec\left(\frac{\pi}{6} + x\right)$ is

(a) $\left(\frac{4}{\sqrt{3}}, \infty\right)$ (b) $\left[\frac{4}{\sqrt{3}}, \infty\right)$

(c) $\left(0, \frac{4}{\sqrt{3}}\right]$ (d) $\left(0, \frac{4}{\sqrt{3}}\right)$

18. Range of the function 'f' defined by $f(x) = \left[\frac{1}{\cos\{x\}}\right]$, where $[\cdot]$ and $\{\cdot\}$ denotes the greatest integer and fractional part function respectively is

(a) set of integers

(b) set of all natural numbers

(c) $\{2, 3, 4, \ldots\ldots\}$ (d) $\{1\}$

19. Given that $f: R^+ \to R, f(x) = |x - 1|$ and $g: [-1, \infty) \to R$, $g(x) = e^x$. If the function is defined, then its domain and range respectively are

(a) $(0, \infty)$ and $[0, \infty)$ (b) $[-1, \infty)$ and $[0, \infty)$

(c) $[-1, \infty)$ and $\left[1 - \frac{1}{e}, \infty\right)$

(d) $[-1, \infty)$ and $\left[\frac{1}{e} - 1, \infty\right)$

20. If 'f' is a real valued function not identically zero, satisfying $f(x + y) + f(x - y) = 2f(x)\, f(y)$, $\forall\ x, y \in R$, then $f(x)$ is

(a) odd (b) even
(c) neither even nor odd
(d) can't say

21. If $f(x) + f(x + a) + f(x + 2a) + \ldots + f(x + na)$ = constant; $\forall\ x \in R$ and $a > 0$ and $f(x)$ is periodic, then period of $f(x)$, is

(a) $(n + 1)a$ (b) $e^{(n+1)a}$
(c) na (d) e^{na}

22. If $F(x)$ and $G(x)$ are even and odd extensions of the functions $f(x) = x\,|x| + \sin|x| + xe^x$, where $x \in (0, 1)$, $g(x) = \cos|x| + x^2 - x$, where $x \in (0, 1)$ respectively to the interval $(-1, 0)$ then $F(x) + G(x)$ in $(-1, 0)$ is

(a) $\sin x + \cos x + xe^{-x}$
(b) $-(\sin x + \cos x + xe^{-x})$
(c) $-(\sin x + \cos x + x + xe^{-x})$
(d) $-(\sin x + \cos x + x^2 + xe^{-x})$

23. Consider the function $f(x) = \text{sgn}(x)$ and $g(x) = x(1 - x^2)$ then which of the following does not hold good?

(a) $(gof)(x)$ is even as well as odd.
(b) $(fog)(x)$ is neither continuous nor differentiable for some x on $(-\infty, \infty)$.
(c) $(fog)(x)$ is neither odd nor even.
(d) $(gof)(x)$ is continuous and differentiable for every x on $(-\infty, \infty)$.

24. If $f(x)$ is an even function and satisfies the relation $x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x)$, where $g(x)$ is an odd function, then $f(10)$ equals

(a) 0 (b) $\frac{50}{79}$ (c) $\frac{49}{75}$ (d) 1

25. Let $f(x) = \dfrac{\sin^{101} x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$, where $[x]$ denotes the integral part of x is

(a) an odd function
(b) an even function
(c) neither odd nor even function
(d) both odd and even function

26. $f(x) = \dfrac{\cos x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$, (where x is not an integral multiple of π and $[\cdot]$ denote the greatest integer function), is

(a) an odd function (b) an even function
(c) neither odd nor even
(d) cann't be determined

27. The period of the function $f(x) = \text{cosec}^2 3x + \cot 4x$ is

(a) $\pi/3$ (b) $\pi/4$ (c) $\pi/6$ (d) π

28. Period of $f(x) = \sin\dfrac{\pi x}{(n-1)!} + \cos\dfrac{\pi x}{n!}$ is

(a) $n!$ (b) $2n!$
(c) $2(n - 1)!$ (d) none of these

29. Let $f: R \to R$ be a periodic function such that $f(T + x) = 1 + [1 - 3f(x) + 3(f(x))^2 - (f(x))^3]^{1/5}$ where T is a fixed positive number, then period of $f(x)$ is

(a) T (b) $2T$
(c) $3T$ (d) none of these

30. If $f(g(x)) = |\cos x|$, $g(f(x)) = \cos^2\sqrt{x}$,then

(a) $f(x)$ is a periodic function and $g(x)$ is a non – periodic function
(b) $f(x)$ is a non – periodic function and $g(x)$ is a periodic function
(c) both $f(x)$ and $g(x)$ are periodic functions
(d) neither $f(x)$ nor $g(x)$ are periodic functions

31. Let $f: R \to R$ be a many one function and there exists a function $g: R \times R \to R$ such that $f(x + y) = g(f(x), y)\ \forall\ x, y \in R$. Then

(a) f is non-periodic
(b) g is non-periodic
(c) f, g may or may not be periodic
(d) none of these

32. For real valued functions $f_i(x) = \sin 2p_i x$, $\forall\ i = 1, 2, 3; p_i \in N$ such that period of $f_1(x) - f_2(x) + 3f_3(x)$ is π, period of $f_1(x) + 2f_2(x)$ is $\dfrac{\pi}{2}$ and period of $f_1(x) + 10f_3(x)$ is $\dfrac{\pi}{3}$ then min $(p_1 + p_2 + p_3)$ is

(a) 5 (b) 10 (c) 11 (d) 8

33. $f(x) = ax + \sin x$, $g(x) = \sin x$, $g(f(x))$ is periodic then a must be
(a) $a \in$ integer
(b) $a \in$ integer and $a > 1$
(c) $a \in$ integer and $1 < a < 3$
(d) none of these

SECTION-II

More than One Correct Answer Type

34. If $[x]$ denotes the greatest integer less than or equal to x, the extreme values of the function $f(x) = [1 + \sin x] + [1 + \sin 2x] + [1 + \sin 3x] + \ldots\ldots + [1 + \sin nx]$, $n \in I^+$, $x \in (0, \pi)$ are
(a) $n - 1$ (b) n (c) $n + 1$ (d) $n + 2$

35. Let $f(x) = \sec^{-1}[1 + \cos^2 x]$, where $[\cdot]$ denotes the greatest integer function, then
(a) the domain of f is R
(b) the domain of f is $[1, 2]$
(c) the range of f is $[1, 2]$
(d) the range of f is $\{\sec^{-1}1, \sec^{-1}2\}$

36. Let $f(x) = [x]^2 + [x + 1] - 3$, where $[x] \le x$. Then
(a) $f(x)$ is many - one and into function
(b) $f(x) = 0$ for infinite number of values of x
(c) $f(x) = 0$ for only two real values
(d) none of these

37. Which of the following functions are periodic?
(a) $f(x) = \sin x + |\sin x|$
(b) $g(x) = \dfrac{(1+\sin x)(1+\sec x)}{(1+\cos x)(1+\operatorname{cosec} x)}$
(c) $h(x) = \max(\sin x, \cos x)$
(d) $p(x) = [x] + \left[x + \dfrac{1}{3}\right] + \left[x + \dfrac{2}{3}\right] - 3x + 10$,
where $[\cdot]$ denotes the greatest integer function

38. Let $f(x) = \dfrac{x}{1+x^2}$ and $g(x) = \dfrac{e^{-x}}{1+[x]}$, where $[\cdot]$ is the greatest integer less than or equal to x then
(a) Domain $(f + g) = R - [-2, 0)$
(b) Domain $(f - g) = R - [-1, 0)$
(c) Range $f \cap$ Range $g = \left[-2, \dfrac{1}{2}\right]$
(d) Range $g = R - \{0\}$

39. If $f : R \to R$ defined as $f(x) = \dfrac{\sin([x]\pi)}{x^2 + x + 1}$, where $[\cdot]$ is the greatest integer less than or equal to x, then
(a) f is one–one (b) f is many–one
(c) f is into (d) f is onto

40. If $f(x) = \cos\left[\dfrac{\pi^2}{2}\right]x + \sin\left[-\dfrac{\pi^2}{2}\right]x$, $[\cdot]$ denoting the greatest integer function then
(a) $f(0) = 1$ (b) $\left(\dfrac{\pi}{3}\right) = \dfrac{1}{\sqrt{3}+1}$
(c) $f(\pi/2) = 0$ (d) $f(\pi) = 0$

41. Which of the following functions is/are periodic?
(a) $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$
(b) $f(x) = \begin{cases} x - [x]; & 2n \le x < 2n + 1 \\ \dfrac{1}{2}; & 2n + 1 \le x < 2n + 2 \end{cases}$, where $[\cdot]$ denotes the greatest integer function
(c) $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]}$, where $[\cdot]$ denotes the greatest integer function
(d) $f(x) = ax - [ax + a] + \tan\left(\dfrac{\pi x}{2}\right)$, where $[\cdot]$ denotes the greatest integer function

42. If $e^x + e^{f(x)} = e$, then for $f(x)$
(a) domain $= (-\infty, 1)$ (b) range $= (-\infty, 1)$
(c) domain $= (-\infty, 0)$ (d) range $= (-\infty, 0)$

43. Let $f(x) = e^{\cos^{-1}\sin(x + \pi/3)}$. Then
(a) $f\left(\dfrac{8\pi}{9}\right) = e^{5\pi/18}$ (b) $f\left(\dfrac{8\pi}{9}\right) = e^{13\pi/18}$
(c) $f\left(-\dfrac{7\pi}{4}\right) = e^{\pi/12}$ (d) $f\left(-\dfrac{7\pi}{4}\right) = e^{11\pi/12}$

44. If $f(x)$ is a polynomial of degree n such that $f(0) = 0, f(1) = \dfrac{1}{2}, \ldots. f(n) = \dfrac{n}{n+1}$, then the value of $f(n + 1)$ is

(a) 1 when n is odd

(b) $\frac{n}{n+2}$ when n is even

(c) $-\frac{n}{n+1}$ when n is odd

(d) -1 when n is even

45. If $f: R \to [-1, \infty)$ and $f(x) = \ln [|\sin 2x| + |\cos 2x|$ (where $[\cdot]$ is the greatest integer function), then

(a) range of f is Z

(b) $f(x)$ is periodic with no fundamental period

(c) $f(x)$ is not invertible in $\left[0, \frac{\pi}{4}\right]$

(d) $f(x)$ is into function

46. The function $f(x) = x^3 + (a + 2)x^2 + 3ax + 5$, $f: R \to R$ is

(a) invertible for infinitely many real values of a

(b) invertible for exactly four integral values of a

(c) invertible for no integral value of a

(d) invertible for $a = 2$

47. If the following functions are defined from $[-1, 1]$ to $[-1, 1]$, identify these which are bijective?

(a) $\sin(\sin^{-1}x)$ (b) $\frac{2}{\pi}\sin^{-1}(\sin x)$

(c) $(\text{sgn } (x) \ln(e^x))$ (d) $x^3 (\text{sgn } (x))$

48. If 'f ' be a differentiable function such that $f(x) = f(2 - x)\ \forall\ x \in R$ and $g(x) = f(1 + x)$, then

(a) $g(x)$ is an odd function

(b) $g(x)$ is an even function

(c) graph of 'f ' is symmetrical w.r.t $x = 1$

(d) $f'(1) = 0$

49. The function 'g' defined by $g(x) = \sin\alpha + \cos\alpha - 1$ where $\alpha = \sin^{-1}\sqrt{\{x\}}$ where { } denotes fractional part of function is

(a) an even function

(b) periodic function

(c) odd function

(d) neither even nor odd

50. The graph of $f: R \to R$ defined by $y = f(x)$ is symmetric with respect to the straight lines $x = a$ and $x = b$. Then which of the following statements is true?

(a) $f(2a - x) = f(x)$ (b) $f(2a + x) = f(-x)$

(c) $f(2b + x) = f(-x)$ (d) f is periodic

51. Let f be a differential function such that $f(x) = f(2 - x)\ \forall\ x \in R$ and $g(x) = f(1 + x)$ then

(a) $g(x)$ is an odd function

(b) $g(x)$ is an even function

(c) Graph of $f(x)$ is symmetrical with line $x = 1$

(d) $f(1) = 0$

52. If $f(x) = -1 + |x - 2|, 0 \le x \le 4$

$g(x) = 2 - |x|, -1 \le x \le 3$, then

(a) $fog(x) = \begin{cases} -1 - x & , -1 \le x \le 0 \\ x - 1 & , 0 < x \le 1 \\ -1 + x & , 1 < x \le 2 \end{cases}$

(b) $fog(x) = \begin{cases} -1 - x & , -1 \le x \le 0 \\ x - 1 & , 0 < x \le 2 \end{cases}$

(c) $gof(x) = \begin{cases} 1 + x & , 0 \le x < 1 \\ 3 - x & , 1 \le x \le 2 \end{cases}$

(d) $gof(x) = \begin{cases} x - 1 & , 2 < x \le 3 \\ 5 - x & , 3 < x \le 4 \end{cases}$

53. Let $x_1, x_2, \ldots\ldots, x_n$ be abscissae of 'n' points where the function 'f' defined as $f(x) = \max\left\{e^{-|x|}, |\sin x|\right\}$ is non-differentiable in the interval $[-10\pi, 10\pi]$. The value of $\sum_{i=1}^{n} x_i$ is equal to

(a) 0 (b) 20π (c) 10π (d) 15π

54. The function

$$f(x) = \cos^{-1}\left(\frac{2[|\sin x| + |\cos x|]}{\sin^2 x + 2\sin x + \frac{11}{4}}\right)$$ is defined if x belongs to (where [.] represents greatest integer function)

(a) $\left[0, \frac{7\pi}{6}\right]$ (b) $\left[0, \frac{\pi}{6}\right]$

(c) $\left[\frac{11\pi}{6}, 2\pi\right]$ (d) $[\pi, 2\pi]$

55. If $2f(x) + xf\left(\frac{1}{x}\right) - 2f\left(\left|\sqrt{2}\sin\pi\left(x + \frac{1}{4}\right)\right|\right)$ $= 4\cos^2\left(\frac{\pi x}{2}\right) + x\cos\frac{\pi}{x}, \forall x \in R - \{0\}$ then which of the following statement(s) is/are true?

(a) $f(2)+f\left(\frac{1}{2}\right)=1$ (b) $f(2)+f(1)=0$

(c) $f(2)+f(1)=f\left(\frac{1}{2}\right)$ (d) $f(1)f\left(\frac{1}{2}\right)f(2)=1$

56. If $f(x+1)+f(x-1)=\sqrt{3}f(x), f(2)=2$ then
(a) $f(4)=-2$ (b) $f(0)=4$
(c) $f(6)=-4$ (d) $f(18)=8$

57. Let $f: X \to Y$ and $g: Y \to Z$. Suppose g is one–one and onto. If $h = gof$, then which of the following is / are true?
(a) If h is one- one, then f is one-one
(b) If f is one-one, then h is one-one and onto
(c) If f is not onto, then h is not onto
(d) If f is not one-one, then h is not one-one

58. If $y = \sin^2 x + \cos^4 x$ then for all real x
(a) the minimum value of y is 2
(b) the minimum value of y is 3/4
(c) $y \le 1$ (d) $y \ge 1/4$

SECTION-III

Comprehension Type

Paragraph for Question No. 59 to 61

Let $f(x)$ be a real valued function such that $(x^2-y^2)f(x)f(y) = xy(f(x+y)(x-y)+(x+y)f(x-y))$ $\forall$ $x, y \in R$ and $f'(0) = 2$.

59. $f(x)$ will be
(a) an even function (b) an odd function
(c) can't say
(d) neither even nor odd

60. $f(\ln 2)$ is
(a) $\frac{5}{3}\ln 3$ (b) $\frac{5}{4}\ln 4$ (c) $\frac{5}{2}\ln 2$ (d) $\ln 4$

61. $\lim_{x\to 0}\left[\frac{f(x)}{\sin x}\right]$ is equal to (where $[\cdot]$ denote greatest integer function)
(a) 1 (b) 2
(c) 3 (d) does not exist

Paragraph for Question No. 62 to 64

For $x \ge 0$, $1, f_1(x) = x$, $f_1(x)=\frac{1}{x}$, $f_3(x) = 1 - x$, $f_4(x)=\frac{1}{1-x}$, $f_5(x)=\frac{x-1}{x}$, $f_6(x)=\frac{x}{x-1}$. This family of functions is closed under compositions i.e. composition of any two of these functions is again one of these.

62. Let F be a function such that $f_1 \, of = f_4$. Then F is equal to
(a) f_1 (b) f_2 (c) f_3 (d) f_4

63. Let G be a function such that $Gof_3 = f_6$. Then F is equal to
(a) f_5 (b) f_4 (c) f_3 (d) f_2

64. Let J be a function such that $f_3 o J o f_2 = f_4$. Then J is equal to
(a) f_6 (b) f_5 (c) f_4 (d) f_3

Paragraph for Question No. 65 to 66

Let f be a real valued function from N to N satisfying the relation $f(m + n) = f(m) + f(n)$ for all $m, n \in N$.

65. If range of f contains all the even numbers, then the value of $f(1)$ is
(a) 1 (b) 2 (c) 1 or 2 (d) 4

66. If domain of f is first '$3m$' natural numbers and if the number of elements common in domain and range is 'm', then the value of $f(1)$ is
(a) 2 (b) 3
(c) 6 (d) can't say

Paragraph for Question No. 67 to 69

Let $f(x) = \log_{\{x\}}[x]$, $g(x) = \log_{[x]}\{x\}$, $h(x) = \log_{\{x\}}\{x\}$ (where $[\cdot]$, $\{\cdot\}$ denotes greatest integer function and fractional part).

67. For $x \in (1, 5)$, $f(x)$ is not defined at how many points?
(a) 5 (b) 4 (c) 3 (d) 2

68. If $A = \{x : x \in \text{domain of } f(x)\}$ and $B = \{x : x \in \text{domain of } g(x)\}$, then $A - B$ will be
(a) (2, 3) (b) (1, 3)
(c) (1, 2) (d) none of these

69. Domain of $h(x)$ is
(a) R (b) I (c) $R - I$ (d) $R^+ - I$

Paragraph for Question No. 70 to 72

If a function $y = f(x); f: A \to B$ then the set A is called as domain of the function and B is called co-domain of the function. For all $x \in A$, the values of y thus obtained comprise the set 'C' where C is called as range of function.

70. The domain of the function $f(x)=\frac{1}{\ln[\cos^{-1}x]}$ is where [·] indicates greatest integer function

(a) $[0, 1]$

(b) $[-1, \cos 2]$

(c) $[-1, \cos 3) \cup (\cos 3, \cos 4)$

(d) $[-1, \cos 3) \cup (\cos 3, \cos 2)$

71. The domain of the function $f(x)=\sqrt{\cos(\sin x)}+\sqrt{\log_x\{x\}}$ where {.} indicates fractional part function

(a) $[1, \pi)$ (b) $(0, 2\pi) - [1, \pi)$

(c) $\left(0,\frac{\pi}{2}\right)-\{1\}$ (d) $(0, 1)$

72. The range of the function of $f(x)=\sin^{-1}\sqrt{x^2+x+1}$ is

(a) $\left[0,\frac{\pi}{2}\right]$ (b) $\left[0,\frac{\pi}{3}\right]$

(c) $\left[\frac{\pi}{3},\frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$

Paragraph for Question No. 73 to 75

Mr. X is a teacher of mathematics. His students want to know the ages of his son's S_1 and S_2. He told that their ages are 'a' and 'b' respectively such that $f(x + y) - axy = f(x) + by^2 \ \forall \ x, y \in R$. After some time students said that information is insufficient, please give more information. Teacher says that $f(1) = 8$ and $f(2) = 32$.

73. The age of S_1 & S_2 will be respectively

(a) 4, 16 (b) 8, 16

(c) 16, 8 (d) 32, 8

74. The function $f(x)$ is

(a) even

(b) odd

(c) neither even nor odd

(d) periodic as well as odd

75. The function $f: R \to R$, then function will be

(a) one-one onto (b) one-one into

(c) many-one onto (d) many-one into

SECTION-IV

Matrix-Match Type

76.

	Column-I		Column-II
(A)	The fundamental period of $f(x) = 3\cos(6\pi x) + 4\sin(6\pi x)$ is	(p)	Does not exist
(B)	The period of the function $f(x) = \sin^4 x + \cos^4 x$ is	(q)	$\frac{1}{3}$
(C)	The period of the function $f(x) = x - [x] + \sin x$	(r)	$\frac{\pi}{2}$

77.

	Column-I		Column-II
(A)	The number of integral solutions of $\frac{x+2}{x^2+1} > \frac{1}{2}$ is	(p)	5
(B)	The number of real values of x satisfying $1 + \lvert e^x - 1\rvert = e^x(e^x - 2)$	(q)	4
(C)	The number of integral solutions of $\lvert[x] - 2x\rvert = 4$ where [·] denotes G.I.F is	(r)	1
(D)	The number of integral solutions satisfying $\lvert\lvert x - 1\rvert - 1\rvert \le 1$	(s)	3

78.

	Column-I		Column-II
(A)	$f: R \to R,\ f(x)=\begin{cases}\frac{x^2}{\lvert x\rvert}, x \neq 0\\ 0, x = 0\end{cases}$	(p)	many-one function
(B)	$f: R \to R,\ f(x) = x - \sin x$	(q)	onto-function
(C)	$f: R \to R, f(x) = (1 + x^2)\,\text{sgn}\,(x)$ [sgn (x) denotes the signum function]	(r)	one-one function
(D)	$f(x) = x^3 + 2x - 7$, $f: Z \to Z$	(s)	into function

79. Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ and $f: A \to B$, then the number of functions 'f' possible for

Column-I		Column-II	
(A)	$i + f(i) < 10\ \forall i \in \{1, 3, 5, 7\}$	(p)	16
(B)	$f(i) - i > 2, \forall i \in \{1, 3, 5, 7\}$	(q)	24
(C)	$f(i) \geq 6, \forall i \in \{1, 3, 5, 7\}$	(r)	0
(D)	$f(i) \neq i + 1, \forall i \in \{1, 3, 5, 7\}$	(s)	81

80. Match the following.

Column-I		Column-II	
(A)	$f(x) = \{x\}$, the fractional part of x	(p)	$f^{-1}(x) = \frac{1}{2}(4^x - 4^{-x})$
(B)	$f(x) = \frac{16^x - 1}{4^x}$	(q)	f is an even function
(C)	$f(x) = \log_4 (x + \sqrt{x^2 + 1})$	(r)	f is a periodic function
(D)	$f(x) = x\frac{3^x - 1}{3^x + 1}$	(s)	f is an odd function

SECTION-V

Integer Answer Type

81. The number of integers belonging to the range of $f(x) = 3\sin\sqrt{\frac{9\pi^2}{16} - x^2}$ is

82. If $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{kr}{1\times3\times5\times7\times9\times.....\times(2r+1)} = 1$ then k^2 =

83. If $f(x) = |x - 2|$ and $g(x) = f(f(x))$, then $\sum_{r=0}^{3} g'(2r-1) =$

84. If $f(x) = x^3 - 12x + P$ and $P \in \{1, 2, 3, 4, 5, ..., 15\}$ and for each 'P', the number of real roots of equation $f(x) = 0$ is denoted by Q then $\frac{1}{5}\Sigma Q$ is equal to

85. If $f(x) = [x] + \sum_{r=1}^{2008} \frac{x + r - [x + r]}{2008}$, then $f(3)$ =

86. Number of natural numbers n satisfying the condition $\left[\sqrt{n+1}\right] \neq \left[\sqrt{n}\right]$ and $1 \leq n \leq 100$ is (where [.] denotes the greatest integer function)

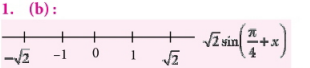
87. Let $f(r)$ denote the number of zeros in $r!$. If $f(m) - f(n) = 3$, then value of $\frac{(m-n)_{\max} - (m-n)_{\min}}{2}$ is

88. If $x^2 + y^2 = 4$, then maximum value of $\left(\frac{x^3 + y^3}{x + y}\right)$ is

89. Let $f(n)$ denote the square of the sum of the digits of natural number n, where $f^2(n)$ denote $f(f(n))$, $f^3(n)$ denote $f(f(f(n)))$ and so on. The value of $\frac{f^{2011}(2011) - f^{2010}(2011)}{f^{2013}(2011) - f^{2012}(2011)} =$

SOLUTIONS

1. (b):



$$f(x) = \left[\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)\right]$$

$$-\sqrt{2} \leq \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \leq \sqrt{2}$$

$$\text{Range of } f = \left[\sqrt{2}\sin\left(\frac{\pi}{4} + x\right)\right] = \{-2, -1, 0, 1\}$$

2. (a): We have $f(x) = \frac{e^x \ln x \, 5^{(x^2+2)}.(x-2)(x-5)}{(2x-3)(x-4)}$

Note that at $x = 3/2$ & $x = 4$ function is not defined and in open interval (3/2,4) function is continuous.

$$\lim_{x\to\frac{3}{2}^+} = \frac{(+ve)(-ve)(-ve)}{(+ve)(-ve)} \to -\infty$$

$$\lim_{x\to4^-} = \frac{(+ve)(+ve)(-ve)}{(+ve)(-ve)} \to \infty$$

In the open interval (3/2,4) the function is continuous & takes up all real values from $(-\infty, \infty)$
Hence range of the function is $(-\infty, \infty)$ or R]

3. (d) : $\ln(\cot^{-1}x) > 0$

$\Rightarrow \cot^{-1}x > 1 \Rightarrow x < \cot 1$

$\therefore$ domain is $(-\infty, \cot 1)$

4. (d) : $f(x) = \dfrac{x^3+x-2}{x^3-1}$

$\Rightarrow f(x) = \left(\dfrac{x-1}{x-1}\right)\left(\dfrac{x^2+x+2}{x^2+x+1}\right) = \dfrac{x^2+x+2}{x^2+x+1}$

when $x \neq 1$

Hence range is $\left(1, \dfrac{7}{3}\right]$.

5. (d) : $-1 \le \cos x \le 1$ $-\sin 1 \le \sin(\cos x) \le \sin x$

$\Rightarrow [\sin(\cos x)] = 0, -1$

$-1 \le \sin x \le 1 \Rightarrow \cos 1 \le \cos(\sin x) \le 1$

$[\cos x(\sin x)] = 0, 1$

$\Rightarrow [\sin(\cos x)] + [\cos(\sin x)] = -1, 0, 1.$

6. (c) : $(f(x))^3 + (x)^3 + (-1)^3 - 3(-1)(x)(f(x)) = 0$

$\Rightarrow (f(x) + x + (-1))(f^2(x) + x^2 + (-1)^2 - xf(x) + x + f(x)) = 0$

[Using $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$]

$f^2(x) + x^2 + (-1)^2 - xf(x) + x + f(x)$

$= \dfrac{1}{2}\{(f(x)-x)^2 + (x+1)^2 + (f(x)+1)^2\} \ge 0$

Equality occurs at $f(x) = x = -1 \Rightarrow f(1) = -1$

But it is given that $f(1) \neq -1$

Only solution is $f(x) = 1 - x$

domain of $\sqrt{1 - f(x)} = \sqrt{x}$ is $x \in R^+ \cup \{0\}$

7. (d) : $f(x) = \dfrac{\tan(\pi[x^2 - x])}{1 + \sin x(\cos x)} = \{0\}$ because of $[x^2 - x]$ is integer.

8. (c) : $f(x) = 0$ if $x \in I$ and for $x \in R \sim I$

$2(x - \{x\}) < 1 + x - [x]$. Thus $f(x) < 1/2$

9. (c) : Domain of $f = \left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right]$

Let $\theta = \sqrt{\dfrac{\pi^2}{9} - x^2}$, Then $0 \le \theta \le \dfrac{\pi}{3}$

and $\tan\theta$ is an increasing function

$\therefore$ Least value of $\tan\theta = \tan 0 = 0$, when $x = \pm\dfrac{\pi}{3}$

and greatest value of $\tan\theta = \tan\dfrac{\pi}{3} = \sqrt{3}$, when $x = 0$

Range of $f = \left[0, 3\sqrt{3}\right]$

10. (b) : $[2\sin x] + [\cos x] = -3$ only if

$[2\sin x] = -2$ and $[\cos x] = -1$

$\therefore \quad -2 \le 2\sin x < -1$ and $-1 \le \cos x < 0$

$\Rightarrow -1 \le \sin x < -\dfrac{1}{2}$ and $-1 \le \cos x < 0$

$\Rightarrow \dfrac{7\pi}{6} < x < \dfrac{11\pi}{6}$ and $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$

Common values of x are given as

$$\frac{7\pi}{6} < x < \frac{3\pi}{2}$$

For these values of x,

$\sin x + \sqrt{3}\cos x = 2\sin\left(\dfrac{\pi}{3} + x\right)$ lies between -2 and -1. Range of $f(x)$ is $(-2, -1)$.

11. (a) : Let $g(x) = \dfrac{x^3 - 3x + 6}{2}$, then $g'(x) = 0$. At maxima/minima the differential coefficient is 0.

$\dfrac{3x^2 - 3}{2} = 0 \Rightarrow x = \pm 1$

Now $g(1) = \dfrac{1 - 3 + 6}{2} = 2$ and $g(-1) = \dfrac{-1 + 3 + 6}{2} = 4$

$\therefore \quad y = 2$ and $y = 4$ touch $y = g(x)$ at $x = 1$ and $x = -1$ respectively.

Now $g(x) = 2 \Rightarrow x^3 - 3x + 6 = 4 \Rightarrow x^3 - 3x + 2 = 0$

$\Rightarrow (x - 1)^2(x + 2) = 0$

also $g(x) = +4 \Rightarrow x^3 - 3x + 6 = 8 \Rightarrow x^3 - 3x - 2 = 0$

$\Rightarrow (x + 1)^2 (x - 2) = 0$

$\therefore \quad g(x) = 2 \Rightarrow x = 1, 1, -2$ and $g(x) = 4$

$\Rightarrow x = -1, -1, 2$

Now due to $(x - \alpha)$ a void will be created in $y = g(x)$ graph. For the given condition to be true that void can be created.

Hence $\alpha \in R - [-2, 2]$

12. (b) : Period of $f(x)$ is 2π, but $f(x)$ is not defined for $x \in (\pi/2, 3\pi/2)$. Hence it suffices to consider $x \in [-\pi/2, \pi/2]$. Further since $f(x)$ is even, we consider $x \in [0, \pi/2]$.

Now $\sqrt{\cos(\sin x)}$ and $\sqrt{\sin(\cos x)}$ are decreasing functions for $x \in [0, \pi/2]$.

$\Rightarrow \quad R_f = [f(\pi/2), f(0)] = \left[\sqrt{\cos 1}, 1 + \sqrt{\sin 1}\right]$

13. (a) : Consider $g(x) = 4x^3 - 8x^2 + 1$

$\Rightarrow g'(x) = 12x^2 - 16x$

Now $g'(x) = 0 \Rightarrow x = 0$ or $x = \dfrac{4}{3} \notin [-1, 1]$

Now $g(-1) = -11, g(0) = 1$ and $g(1) = -3$
Since $\cos x \in [-1, 1]$, the range of $f(x)$ is $[-11, 1]$.

14. (b) : Given $a^2 + b^2 + c^2 = 1$
Now $(a + b + c)^2 \geq 0$
i.e. $a^2 + b^2 + c^2 + 2(ab + bc + ca) \geq 0$

i.e. $1 + 2(ab + bc + ca) \geq 0$ *i.e.* $ab + bc + ca \geq -\frac{1}{2}$

Also $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$
i.e. $2(a^2 + b^2 + c^2) - 2(ab + bc + ca) \geq 0$
$\therefore \quad 1 - (\Sigma ab) \geq 0$ *i.e.* $\Sigma ab \leq 1$

Hence Σab lies in $\left[-\frac{1}{2}, 1\right]$

15. (d) : The domain excludes the roots of
$x^2 + 3x + 2 = 0$ *i.e.*, $-1, -2$
Further $x + 3 > 0 \Rightarrow x > -3$
The domain is $(-3, \infty) - \{-1, -2\}$

16. (a) : $|b| < 2$ and $2x^4 + bx^3 - 6x^2 - 4bx - 8 = (x^2 - 4)(2x^2 + bx + 2)$

17. (b) : $a, b > 0 \Rightarrow \frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$

$$\Rightarrow \quad f(x) \geq \frac{2}{\sqrt{\cos\left(\frac{\pi}{6} - x\right)\cos\left(\frac{\pi}{6} + x\right)}} \geq \frac{2}{\sqrt{\frac{1}{4} + \frac{\cos 2x}{2}}}$$

$$0 \leq x \leq \frac{\pi}{3} \Rightarrow \frac{-1}{2} \leq \cos 2x \leq 1$$

$$\Rightarrow \quad 0 \leq \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \leq \frac{\sqrt{3}}{2} \Rightarrow f(x) \geq \frac{4}{\sqrt{3}}$$

Since 'f' is continuous. Range of 'f' is $\left[\frac{4}{\sqrt{3}}, \infty\right)$

18. (d) : $0 \leq \{x\} < 1 \Rightarrow \cos 1 < \cos\{x\} \leq \cos 0$

$$\Rightarrow \quad 1 \leq \frac{1}{\cos\{x\}} < \frac{1}{\cos 1} < 2 \Rightarrow \left[\frac{1}{\cos\{x\}}\right] = 1$$

$\Rightarrow$ Range of the function is $\{1\}$

19. (b) :

$$f(x) = |x - 1| = \begin{cases} 1 - x, & 0 < x < 1 \\ x - 1, & x \geq 1 \end{cases}, g(x) = e^x, x \geq -1$$

$$fog(x) = \begin{cases} 1 - g(x), & 0 < g(x) < 1 \text{ is } -1 \leq x < 0 \\ g(x) - 1, & g(x) \geq 1 \text{ is } 0 \leq x \end{cases}$$

$$= \begin{cases} 1 - e^x, & -1 \leq x < 0 \\ e^x - 1, & x \geq 0 \end{cases}$$

$\therefore$ Domain $= [-1, \infty)$
fog is decreasing in $[-1, 0]$ and increasing in $(0, \infty)$
$x \to \infty \Rightarrow fog(x) \to \infty \Rightarrow fog(0) = 0$. Range $= [0, \infty)$

20. (b) : $f(x + y) + f(x - y) = 2f(x)\, f(y)$
$f(0) = 2f^2(0)$ and $f(0) \neq 0$, (f is given to be not identically to 0) $f(0) = 1$
$f(y) + f(-y) = 2f(0)\, f(y)$
$\Rightarrow \quad f(y) + f(-y) = 2f(y)$
$\Rightarrow \quad f(-y) = f(y)$ similarly $f(-x) = f(x)$, even.

21. (a): $f(x) + f(x + a) + f(x + 2a) + \ldots + f(x + na) = k$ …(i)

Replacing $x = x + a$, we get
$f(x + a) + f(x + 2a) + \ldots + f(x + (n + 1)a) = k$ …(ii)
Subtracting (i) and (ii), we get
$f(x) - f(x + (n + 1)a) = 0$
$\Rightarrow \quad f(x) = f(x + (n + 1)a \qquad \Rightarrow T = (n + 1)\, a$

22. (c) : $f(x) = f(-x)$ where $f(x) = x^2 + \sin x + xe^x$
$F(x) = x^2 - \sin x - xe^{-x}$ …(1)
$g(x) = -g(-x)$, where $g(x) = \cos x + x^2 - x$
$G(x) = -(\cos x + x^2 + x) = -\cos x - x^2 - x$
$\therefore \quad F(x) + G(x) = -\sin x - xe^{-x} - \cos x - x$
$= -(\sin x + \cos x + x + xe^{-x})$

23. (c) : $f(x) = \text{sgn}(x)$, $g(x) = x(1 - x^2)$
$\therefore \quad (gof)(x) = 0$ and

$$(fog)(x) = \begin{cases} 1, & \text{if } x \in (-\infty, -1) \cup (0, 1) \\ 0, & \text{if } x = 1 (\text{or}) -1 (\text{or}) 0 \\ -1, & \text{if } x \in (-1, 0) \cup (1, \infty) \end{cases}$$

In general $fog(x)$ is not equal to $gof(x)$.

24. (a) : $f(-x) = f(x)$, $g(-x) = -g(x)$

$$\therefore \quad x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x)$$

and $x^2 f(-x) - 2f\left(\frac{-1}{x}\right) = g(-x)$

$$\Rightarrow \quad x^2 f(x) - 2f\left(\frac{1}{x}\right) = -g(x)$$

$$\Rightarrow \quad x^2 f(x) - 2f\left(\frac{1}{x}\right) = 0$$

repalce x with $\frac{1}{x}$, we get $\frac{1}{x^2} f\left(\frac{1}{x}\right) - 2f(x) = 0$

$\therefore \quad -3x^2 f(x) = 0 \Rightarrow f(x) = 0 \quad \therefore f(10) = 0 \ (\because x \neq 0)$

25. (b) : When $x \neq n\pi$, $n \in I$, $\sin x = 0$
and $\left[\frac{x}{\pi}\right] + \frac{1}{2} \neq 0$

$\therefore \quad f(x) = 0$

$\therefore$ When $x = n\pi$, $f(x) = 0$ and $f(-x) = 0$

$\therefore \quad f(-x) = f(x)$

When $x \neq n\pi$, $n \in I$, $\frac{x}{\pi} \neq$ an integer

$$\therefore \left[\frac{x}{\pi}\right] + \left[-\frac{x}{\pi}\right] = -1 \ \therefore \ \left[-\frac{x}{\pi}\right] = -1 - \left[\frac{x}{\pi}\right]$$

$$\Rightarrow \left[-\frac{x}{\pi}\right] + \frac{1}{2} = -\left[\frac{x}{\pi}\right] - \frac{1}{2} = -\left(\left[\frac{x}{\pi}\right] + \frac{1}{2}\right)$$

$$\text{Now } f(-x) = \frac{\sin^{101}(-x)}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}} = \frac{-\sin x}{-\left(\left[\frac{x}{\pi}\right] + \frac{1}{2}\right)} = \frac{\sin x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}}$$

$= f(x)$

Hence in all cases $f(-x) = f(x)$

26. (a) : $$f(-x) = \frac{\cos(-x)}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}} = \frac{\cos x}{-\left[\frac{x}{\pi}\right] - 1 + \frac{1}{2}}$$

$$\left(\text{as } x \neq n\pi \Rightarrow \frac{x}{\pi} \notin I, \text{ so as} \left[-\frac{x}{\pi}\right] = -\left[\frac{x}{\pi}\right] - 1\right)$$

$$= -\frac{\cos x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} = -f(x) \Rightarrow f(x) \text{ is an odd function.}$$

27. (d) : $f(x) = 2/(1 - \cos 6x) + 1/(\tan 4x)$. So the period of f = l.c.m (period of $\cos 6x$, period of $\tan 4x$) = l.c.m $(\pi/3, \pi/4) = \pi$.

28. (b) : $\sin\frac{\pi x}{(n-1)!}$ is a periodic function with

$$\text{period } 2\pi + \frac{\pi}{(n-1)!} = 2(n-1)!$$

$$\text{Period of } \cos\frac{\pi x}{n!} = 2\pi + \frac{\pi}{n!} = 2n!$$

L. C. M. of $2(n-1)!$ and $2\,(n!)$

$\therefore$ Period of $f(x)$ is $2\,(n!)$

29. (b) : Given: $f(T + x) = 1 + (1 - f(x))^3]^{1/3}$

$= 1 + (1 - f(x))$

$\Rightarrow \quad f(T + x) + f(x) = 2 \qquad \ldots(i)$

$\Rightarrow \quad f(2T + x) + f(T + x) = 2 \qquad \ldots(ii)$

Subtracting (i) and (ii), we get

$\Rightarrow \quad f(2T + x) = f(x)$

Also T is positive and least therefore period of $f(x) = 2T$

30. (b) : Given, $f(g(x)) = |\cos x| = \sqrt{\cos^2 x} \qquad \ldots(1)$

$g(f(x)) = \cos^2\sqrt{x} \qquad \ldots(2)$

From (1) and (2), $f(x) = \sqrt{x}$ and $g(x) = \cos^2 x$

Clearly $f(x)$ is a non-periodic function and $g(x)$ is a periodic function

31. (d) : Since f is many one. Let $f(\alpha) = f(\beta)$ where $\alpha \neq \beta$

Now $f(\alpha + y) = g(f(\alpha), y) = g(f(\beta), y) = f(\beta + y)$

$\therefore \quad g$ is periodic

also for $y = t - \alpha \Rightarrow \ f(t) = f(t + \beta - \alpha) \ \forall \ t \in R$

$\therefore \quad f$ is periodic

32. (c) : p must be divisible by 2 and 3 both

Hence $p_1 = 6$, $p_2 = 2$, $p_3 = 3$.

33. (b) : $g(f(x)) = \text{sing}(ax + \sin x)$

$\Rightarrow \ a$ must be integer and $a > 1$.

34. (b, c) : We have

$f(x) = [1 + \sin x] + [1 + \sin 2x] + [1 + \sin 3x] + \ldots.. + [1 + \sin nx]$

$= 1 + [\sin x] + 1 + [\sin 2x] + 1 + [\sin 3x] + \ldots.+ 1 + [\sin nx]$

$= n + [\sin x] + [\sin 2x] + [\sin 3x] + \ldots.. + [\sin nx]$

$= n + 1$ (if n is odd)

$\because \quad x \in (0, \pi) \quad \therefore \quad \sin x > 0$

35. (a, d) : $\therefore f(x) = \sec^{-1}[1 + \cos^2 x]$

Domain of $f(x)$ is R and $\therefore \ 0 \leq \cos^2 x \leq 1$

$\Rightarrow \ 1 \leq (1 + \cos^2 x) \leq 2$

$\therefore \quad [1 + \cos^2 x] = 1$ and 2

Then, range of $f(x)$ is $\sec^{-1} 1$ and $\sec^{-1} 2$

$\Rightarrow \ \{\sec^{-1} 1, \sec^{-1} 2\}$

36. (a, b) : $f(x) = [x]^2 + [x + 1] - 3 = [x]^2 + 1 + [x] - 3$

$= ([x] + 2)([x] - 1)$; $f(x) = 0$, then $[x] = -2$

$\Rightarrow \ -2 \leq x < -1$

and $[x] = 1$; $\Rightarrow \ 1 \leq x < 2$ many one and into

37. (a, b, c, d) : Option (a): $\because$ Periods of $\sin x$ and $|\sin x|$ are 2π and π respectively.

$\therefore$ period of $f(x)$ = LCM of $\{2\pi, \pi\} = 2\pi$

Option (b): Periods of $\sin x$, $\cos x$, $\text{cosec}\ x$ are 2π hence, period of $g(x) = 2\pi$

Option (c): Period of $h(x)$ = LCM $\{2\pi, 2\pi\} = 2\pi$

Option (d):

$$p(x) = [x] + \left[x + \frac{1}{3}\right] + \left[x + \frac{2}{3}\right] - \left(x + \left(x + \frac{1}{3}\right) + \left(x + \frac{2}{3}\right)\right) + 11$$

$$= 11 - \{x\} - \left\{x + \frac{1}{3}\right\} - \left\{x + \frac{2}{3}\right\}$$

$\therefore$ period of $p(x)$ is 1.

38. (b, d): Domain of $f = R$, Domain $g = R - [-1, 0)$; because $-1 \le x < 0$ thus $1 + [x] = 0$

$\therefore$ Domain of $f - g = R - [-1, 0)$; since $e^{-x} > 0$

$\Rightarrow (1 + [x])y > 0$

$\Rightarrow y > 0$ is $1 + [x] > 0$, $\Rightarrow y < 0$ is $1 + [x] < 0$

$\Rightarrow y \in R - \{0\}$

39. (b, c): $f(x) = \dfrac{\sin([x]\pi)}{x^2 + x + 1} = 0, \forall x \in R$

$\Rightarrow f$ is constant function

$\Rightarrow f$ is many one and into function

40. (a, b, c): $f(x) = \cos\left[\dfrac{\pi^2}{2}\right]x + \sin\left[-\dfrac{\pi^2}{2}\right]x$

$\left[\dfrac{\pi^2}{2}\right] = 3$ & $\left[-\dfrac{\pi^2}{2}\right] = -4$

$f(x) = \cos 3x - \sin 4x$

$f(0) = 1$ similarly others.

41. (a): (a) $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

$\Rightarrow \quad f(x+k) = \begin{cases} 1, & x + k \text{ is rational} \\ 0, & x + k \text{ is irrational} \end{cases}$

where k is any rational number

$f(x+k) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

$\Rightarrow f(x + k) = f(x)$

$\Rightarrow f(x)$ is periodic function, but its fundamental period cannot be determined

(b) $f(x) = \begin{cases} x - [x], & 2n \le x < 2n + 1 \\ 1/2, & 2n + 1 \le x < 2n + 2 \end{cases}$

(c) $f(x) = (-1)^{\left[\frac{2x}{\pi}\right]} \Rightarrow f(x + \pi)$

$= (-1)(-1)^{\left[\frac{2(\pi + x)}{\pi}\right]} = (-1)(-1)^{\left[\frac{2x}{\pi}\right] + 2} = (-1)(-1)^{\left[\frac{2x}{\pi}\right]}$

Hence, the period is π.

(d) $f(x) = ax + a - [ax + a] + \tan\left(\dfrac{\pi x}{2}\right) = \{ax + a\} + \tan\left(\dfrac{\pi x}{2}\right)$

$\{ax + a\}$ is periodic with period $1/a$. $\tan\left(\dfrac{\pi x}{2}\right)$ is periodic with period 2.

Now, the L.C.M. of $1/a$ and 2 is 2. Hence, the period of $f(x)$ is 2.

42. (a, b): We have $e^{f(x)} = e - e^x$

$\Rightarrow \quad e - e^x > 0 \quad \Rightarrow \quad e > e^x \quad \Rightarrow x < 1$

43. (b, c):

$f\left(\dfrac{8\pi}{9}\right) = e^{\cos^{-1}(\sin 8\pi/9 + \pi/3)} = e^{\cos^{-1}(\sin 11\pi/9)}$

$= e^{\cos^{-1}(\cos 13\pi/18)} = e^{13\pi/18}$

$f\left(-\dfrac{7\pi}{4}\right) = e^{\cos^{-1}\left(\sin(-7\pi/4 + \pi/3)\right)}$

$= e^{\cos^{-1}\left(\sin(-17\pi/12)\right)} = e^{\cos^{-1}(\cos \pi/12)} = e^{\pi/12}$

44. (a, b): $(x + 1)f(x) - x = 0$ is $(n + 1)^{\text{th}}$ degree

$\Rightarrow (x + 1)f(x) - x = (x - 0)(x - 1)(x - 2).....(x - n)K$ (Identity)

put $x = n + 1$

$\Rightarrow (n + 2)f(n + 1) - (n + 1) = (n + 1)!K$ ...(1)

put $x = -1 \Rightarrow 1 = (-1)n + 1(n + 1)!$ ...(2)

From (1) and (2), we get

$\Rightarrow f(n+1) = \dfrac{(n+1) + (-1)^{n+1}}{n+2} = \begin{cases} 1, & n \text{ odd} \\ \dfrac{n}{n+2}, & n \text{ even} \end{cases}$

45. (b, c, d): Range of $|\sin 2x| + |\cos 2x|$ is $[1, \sqrt{2}]$

$[|\sin 2x| + |\cos 2x|] = 1$

$\therefore \quad f(x) = 0 \Rightarrow f(x)$ is a constant function

$\therefore \quad f(x)$ is periodic with no fundamental period.

Clearly $f(x)$ is into function and f is not invertible.

46. (a, b, d): $f'(x) = 3x^2 + (a + 2)2x + 3a$

$f' \ge 0 \ \forall x \in R \Rightarrow \Delta \le 0 \Rightarrow 4(a + 2)^2 - 36a \le 0$

$\Rightarrow a^2 - 5a + 4 \le 0 \Rightarrow (a - 1) - (a - 4) \le 0$

$\Rightarrow 1 \le a \le 4$

47. (a): (a) $f(x) = \sin(\sin^{-1}x) = x, \forall x \in [-1, 1]$ which is one – one and onto

(b) $f(x) = \dfrac{2}{\pi}\sin^{-1}(\sin x) = \dfrac{2}{\pi}x$

The range of the function for $x \in [-1\ 1]$ is $\left[-\dfrac{2}{\pi}, \dfrac{2}{\pi}\right]$ which is a subset of $[-1, 1]$.

Hence the function is one-one but not onto. Hence not bijective

(c)

$f(x) = (\text{Sgn}(x))\ \log(e^x) = (\text{Sgn}(x))\ x = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}$

This function has the range $[0, 1]$ which is a subset of $[-1, 1]$. Hence the function is into. Also the function is many one.

(d) $f(x)=x^3\,\text{sgn}(x)=\begin{cases}x^3, & x>0\\ -x^3, & x<0\\ 0, & x=0\end{cases}$ which is many one and into

48. (b, c, d) : $f(x)=f(2-x)$

$\Rightarrow f(x+1)=f(1-x)$ [Replace x by $x+1$]

$\Rightarrow$ (c) is true

also $g(x)=f(x+1)$

$\Rightarrow g(-x)=f(-x+1)=f(x+1)=g(x)$

$\Rightarrow$ (b) is true

Now differentiating $f(x+1)=f(1-x)$

$\Rightarrow f'(1+x)=-f'(1-x)$

put $x=0 \Rightarrow f'(1)=0$

49. (a, b) : $g(x)=\sin(\sin^{-1}\sqrt{\{x\}}) + \cos(\sin^{-1}\sqrt{\{x\}})-1$

$=\sqrt{\{x\}}+\cos(\cos^{-1}\sqrt{1-\{x\}})-1$

$=\sqrt{\{x\}}+\sqrt{1-\{x\}}-1$

If $x\in I$ then $\{x\}=0 \Rightarrow g(x)=0$

$\Rightarrow g(-x)=g(x)\Rightarrow g$ is even

If $x\notin I$ then $\{-x\}=1-\{x\}$

$\Rightarrow g(-x)=\sqrt{1-\{x\}}+\sqrt{\{x\}}-1=g(x)$

$\Rightarrow$ 'g' is even function

$g(x)=\begin{cases}0, & x\in I\\ g(-x), & x\notin I\end{cases}\Rightarrow$ 'g' is periodic function

50. (a, b, c, d) : Given $f(a+x)=f(a-x)$

and (a) $f(2a-x)=f(a+(a-x))=f(a-(a-x))=f(x)$

(b) $f(2a+x)=f(a+(a+x))=f(a-(a+x))=f(-x)$

(c) $f(2b+x)=f(-x)$

(d)qWe have $f(2a+x)=f(2b+x)$

$\therefore\ 2b-2a$ is a period of f.

51. (b, c, d)

52. (b, c, d) : $f[g(x)]=\begin{cases}1-g(x) & ,\ 0\le g(x)\le 2\\ g(x)-3 & ,\ 2\le g(x)\le 3\end{cases}$

$=\begin{cases}1-(2+x) & 0\le 2+x\le 2; \quad -1\le x\le 6\\ 2+x-3 & 2<2+x\le 4; \quad -1\le x\le 0\\ 1-(2-x) & 0\le 2-x\le 2; \quad 0<x\le 3\\ 2-(x-3) & 2<2-x\le 4; \quad 0<x\le 3\end{cases}$

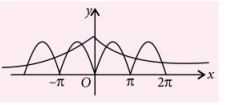
$=\begin{cases}-1-x & -1\le x\le 0\\ x-1 & x\in\phi\\ -1+x & 0<x\le 2\\ -x-1 & x\in\phi\end{cases}$

and $(gof)\,x=g\{f(x)\}=\begin{cases}2+f(x) & ,\ -1\le f(x)\le 0\\ 2-f(x) & ,\ 0<f(x)\le 3\end{cases}$

$=\begin{cases}2+1-x & -1\le 1-x\le 0 \quad \text{and} \quad 0\le x\le 2\\ 2-(1-x) & 0<1-x\le 3 \quad \text{and} \quad 0\le x\le 2\\ 2+x-3 & -1\le x-3\le 0 \quad \text{and} \quad 2<x\le 4\\ 2(x-3) & 0<x-3\le 3 \quad \text{and} \quad 2<x\le 4\end{cases}$

$=\begin{cases}1+x, & 0\le x<1\\ 3-x, & 1\le x\le 2\\ x-1, & 2<x\le 3\\ 5-x, & 3<x\le 4\end{cases}$

53. (a) :



Here we observe that there are two points in $(0,\pi)$ and two points in $(-\pi,0)$ where $f(x)$ is not differentiable.

Consequently $\sum_{i=1}^{n}x_i=0$ in $[-10\pi,10\pi]$

54. (a, b, c) : $1\le|\sin x|+|\cos x|\le\sqrt{2}$

$[|\sin x|+|\cos x|]=1$

$f(x)$ is defined if $\sin^2 x+2\sin x+\frac{11}{4}\ge 2$

$\Rightarrow\ (\sin x+1)^2\ge\frac{1}{4}$

$\Rightarrow\ \sin x+1\ge\frac{1}{2}$ or $\sin x+1\le-\frac{1}{2}$

$\Rightarrow\ \sin x\ge-\frac{1}{2}$ or $\sin x\le-\frac{3}{2}$ (which is not true)

55. (a, b, c) : Replace x by 2, we get

$2f(2)+2f\left(\frac{1}{2}\right)-2f(1)=4\Rightarrow f(2)+f\left(\frac{1}{2}\right)=2+f(1)$ …(1)

Replace x by 1, $f(1)=-1$ …(2)

Replace x by $\frac{1}{2}$, $2f\left(\frac{1}{2}\right)+\frac{1}{2}f\left(\frac{1}{2}\right)+2=\frac{5}{2}$

Solving (1) & (3) we get $f\left(\frac{1}{2}\right)=0; f(2)=1$

56. (a, b, c) : Replace x by $x+1$ then equation becomes $f(x+2)-\sqrt{3}f(x+1)+f(x)=0$

$=(E^2-\sqrt{3}E+1)f(x)=0$

Roots of $E^2-\sqrt{3}E+1=0$ are $\text{cis}\frac{\pi}{6}, \text{cis}-\frac{\pi}{6}$

$f(2)=A\,\text{cis}\frac{\pi x}{6}+B\,\text{cis}\left(\frac{-\pi x}{6}\right)$

$f(z)=2=\frac{A+B}{2}+\frac{i}{2}(A-B)$

$\therefore\ A=B=2$

$\therefore\ f(x)=4\cos\frac{\pi x}{6}$

57. (a, c, d) : (a) is correct,

(b) f, g are one-one $\Rightarrow h=gof$ is one-one.

But h need not be onto. Consider $X=\{1,2\}$, $Y=\{a,b,c\}$ and $Z=\{p,q,r\}; f=\{(1,a),(2,b)\}$ and $g=\{(a,p),(b,p),(c,r)\}$

Then f is one-one and g is one-one and onto.

But $h=\{(1,p),(2,q)\}$ is not onto.

(c) is true. If h is onto, then $y\in Y$

$\Rightarrow\ g(y)\in Z\ \Rightarrow\ h(x)=g(y)$ for some $x\in X$

$\Rightarrow\ (gof)(x)=g(y)$ for some $x\in X$

$\Rightarrow\ f(x)=y$ for some $x\in X$

$\therefore\ f$ is onto.

58. (b, c) : $y=\sin^2x+\cos^2x\,(1-\sin^2x)$

$=1-\sin^2x\cos^2x=1-\frac{1}{4}\sin^2 2x$

$\therefore$ Minimum of $y=1-\frac{1}{4}=\frac{3}{4}$

Maximum of $y=1-0=1$

59. (b) : Dividing throughout by $xy(x^2-y^2)$, we get

$\frac{f(x)f(y)}{x.y}=\frac{f(x+y)}{x+y}+\frac{f(x-y)}{x-y}$

Let $F(x)=\frac{f(x)}{x}$

Then $F(0)=\lim_{x\to0}\frac{f(x)}{x}=\frac{f'(0)}{1}=2$

$F(x)F(y)=F(x)+F(y)$

put $y=0$

$F(x)F(0)=F(x)+F(0)$

$\Rightarrow\ 2F(x)=F(x)+2$

$\Rightarrow\ F(x)=2x\ \Rightarrow\ f(x)=2x$

$\therefore\ f(x)$ is an odd function

60. (d) : $f(\ln 2)=2\ln 2$

61. (b) : $\lim_{x\to0}\left[\frac{f(x)}{\sin x}\right]=\lim_{x\to0}\left[\frac{2x}{\sin x}\right]=2$

62. (a) : $f_1oF=f_4\Rightarrow F=f_1^{-1}of_4=f_1of_4\Rightarrow F=f_4$

63. (a) : $Gof_3=f_6\ \Rightarrow\ G=f_6of_3^{-1}=f_6of_3$

so $G(x)=f_6(1-x)=\frac{1-x}{1-x-1}=1-\frac{1}{x}=f_5(x)$

64. (c) : $J=f_3^{-1}of_4of_2^{-1}=f_3of_4of_2$

$\therefore\ J(x)=f_3of_4\left(\frac{1}{x}\right)=f_3\left(\frac{1}{1-\frac{1}{x}}\right)$

$=f_3\left(\frac{x}{x-1}\right)=1-\left(\frac{x}{x-1}\right)=f_4(x)$

65. (c) : $f(m+n)=f(m)+f(n)$

$m=1, n=N-1$

$f(N)=f(1)+f(N-1)$

$f(N-1)=f(1)+f(N-2)$

$\vdots\quad\vdots\quad\vdots$

$f(N)=Nf(1)$

If $f(1)=1\ \Rightarrow\ f(N)=N$ which contains even numbers.

If $f(1)=2\ \Rightarrow f(N)=2N$ which contains even numbers.

66. (b) : $f(n)=nf(1)$

Number of elements common in domain and range is m.

$n(R\cap D)=\frac{n(D)}{3}$

67. (c) : $\{x\}\neq0\ \Rightarrow\ x\neq2,3,4$

68. (c) : $[x]\geq2\Rightarrow x\in[2,0)$

$A=\{x: x\notin I^+\}$

$B=\{x: x\notin I^+\cap(2,\infty)\}$

$A-B=(1,2)$

69. (c) : $\{x\}\neq0\ \Rightarrow\ x\in R-I$

70. (b) : Case I : $[\cos^{-1}x]\neq1$

$\cos^{-1}x < 0$ $\cos^{-1}x \geq 2$

$x > 1$ or $x \leq \cos 2$

Case II : $[\cos^{-1}x] > 0$

$-1 \leq x \leq 1$, $\cos^{-1}x \geq 1$

$x < \cos 1 \Rightarrow$ Domain of $f(x) = [-1, \cos 2]$

71. (d) : $\cos(\sin x)$ is defined for all real x.

$\log_x \{x\} \geq 0$

Case I : $0 < x < 1$

$\{x\} \leq 1$; $x \in (0, 1)$

Case II : $x > 1$ and $\{x\} > 0$

$\{x\} \geq 1$ but $0 < \{x\} < 1$ is not possible.

72. (c) : Least value of $x^2 + x + 1 = 3/4$ (at $x = 1/2$)

$\therefore$ least value of $\sqrt{x^2+x+1} = \dfrac{\sqrt{3}}{2}$

Also $\sqrt{x^2+x+1} \leq 1$

Thus $\dfrac{\sqrt{3}}{2} \leq \sqrt{x^2-x+1} \leq 1$

$\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) \leq \sin^{-1}\sqrt{x^2+x+1} \leq \sin^{-1}(1)$

$\dfrac{\pi}{3} \leq \sin\sqrt{x^2+x+1} \leq \dfrac{\pi}{2}$

Range of $f = \left[\dfrac{\pi}{3}, \dfrac{\pi}{2}\right]$

73. (c) : Given $f(1) = 8$ and $f(2) = 32$

putting $x = 1$ and $y = 1$

$f(1 + 1) - a = f(1) + b$

$f(2) = a + f(1) + b = a + b + 8$ or $32 = a + b + 8$

$\therefore \quad a + b = 24 \qquad \ldots$ (i)

putting $x = 2$, $y = -1$

$f(1) + 2a = f(2) + b$ or $8 + 2a = 32 + b$

$2a - b = 24 \qquad \ldots$ (ii)

from (i) & (ii), we get

$a = 16$, $b = 8$

74. (a) : $f(x + y) - 16xy = f(x) + 8y^2$

$f(x + y) = 16xy + f(x) + 8y^2$

$f(x + y) - f(x) = 16xy + f(x) + 8y^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{16xh+8h^2}{h} = 16x$$

$\therefore \quad f(x) = 8x^2$

75. (d) : The function will be many one into function.

76. $\mathbf{A \to q; B \to r; C \to p}$

(A) $f(x) = \dfrac{1}{5}\sin(6\pi x + \alpha)$ whose range is $\dfrac{2\pi}{(6\pi)} = \dfrac{1}{3}$

(B) Period of $f(x)$ is $\dfrac{\pi}{2}$

(C) Period doesnot exist

77. $\mathbf{A \to s; B \to r; C \to q; D \to p}$

(A) $\dfrac{x+2}{x^2+1} > \dfrac{1}{2} \Rightarrow 2x + 4 > x^2 + 1$

$\Rightarrow x^2 - 2x - 3 < 0 \Rightarrow -1 < x < 3 \Rightarrow x = 0, 1, 2$

(B) If $e^x > 1 \Rightarrow 1 + e^x - 1 = e^{2x} - 2e^x$

$\Rightarrow e^{2x} - 3e^x = 0 \Rightarrow e^x = 3 \Rightarrow x = \log_e 3$

If $e^x < 1 \Rightarrow 1 - e^x + 1 = e^{2x} - 2e^x$

$\Rightarrow e^{2x} - e^x - 2 = 0$

$\Rightarrow (e^x - 2)(e^x + 1) = 0$

$\Rightarrow x = \log_e 2$ not possible.

(C) $x = \pm\dfrac{7}{2}$, $x = \pm 4$ are four solutions.

(D) $-1 \leq |x - 1| - 1 \leq 1 \Rightarrow 0 \leq |x - 1| \leq 2$

$-2 \leq x - 1 \leq 2 \Rightarrow -1 \leq x \leq 3$

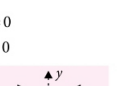
$\Rightarrow x = -1, 0, 1, 2, 3$

There are 5 solutions.

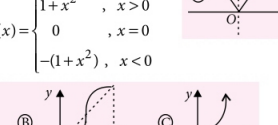
78. $\mathbf{A \to p, s; B \to q, r; C \to r, s; D \to r, s}$

(A) $f(x) = \begin{cases} x & , x > 0 \\ -x & , x < 0 \\ 0 & , x = 0 \end{cases} = \begin{cases} |x|; x \neq 0 \\ 0; x = 0 \end{cases}$

(B) $f(x) = x - \sin x$.



(C) $f(x) = \begin{cases} 1 + x^2 & , x > 0 \\ 0 & , x = 0 \\ -(1 + x^2) & , x < 0 \end{cases}$



(D) Its a cubic but on set of integers, hence one one & into.

79. $\mathbf{A \to q; B \to r; C - p; D \to s}$

(A) For 1, 3, 5, $7 \in A$ we have 4, 3, 2, 1, choices respectively.

(B) Image of $f(7)$ should be greater than 9.

(C) For 1, 3, 5, 7 $\in A$ we have 2, 2, 2, 2 choices respectively.
(D) for any $i \in A$, we have 3 choices.

80. $\mathbf{A \to r; B \to s; C \to p, s; D \to q}$
(A) $f(x) = x - [x]$ is periodic function
(B) $f(x) = 4^x - 4^{-x}$ is an odd function
(C) $y = \log_4(x + \sqrt{x^2+1}) \Rightarrow 4^y = x + \sqrt{x^2+1}$
y is an odd function.
(D) $f(x) = x\dfrac{3^x - 1}{3^x + 1}$ is an even function.

81. (4) : $|x| \le \dfrac{3\pi}{4}$. Number of integers of range = 4

82. (4) :

$$T_r = \frac{1}{2}\left(\frac{1}{1\times3\times5\times.....\times2r-1} - \frac{1}{1\times3\times5\times...\times2r+1}\right)$$

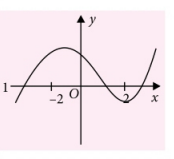
$$\sum_{r=1}^{n} T_r = \frac{1}{2}\left(1 - \frac{1}{1\times3\times5\times...\times2n+1}\right)$$

$\therefore \quad \dfrac{k}{2} = 1 \Rightarrow k^2 = 4$

83. (0) : $g(x) = \big||x-2|-2\big| = \begin{cases} |x|, & x \le 2 \\ 4-x, & 2 < x \le 4 \\ x-4, & 4 < x \end{cases}$

$g'(-1) + g'(1) + g'(3) + g(5) = -1 + 1 - 1 + 1 = 0$

84. (9) : $f(x) = x^3 - 12x + P$
$\Rightarrow f'(x) = 3x^2 - 12$
$= 3(x-2)(x+2)$
$f'(x)$ is positive
$f(x)$ increases in
$(-\infty, -2) \cup (2, \infty)$
$f(x)$ decreases in $(-2, 2)$
$\because f(x)$ is continuous



$$\left.\begin{matrix} f(-2) = P + 16 \\ f(2) = P - 16 \end{matrix}\right\} \forall P \in \{1, 2, ..., 15\}$$

$f(-2) > 0, \ f(2) < 0$
If has 3 roots in each case
$\therefore \quad \Sigma Q = 3 \times 15 = 45$

85. (3) : $f(x) = [x] + \sum_{k=1}^{2008} \dfrac{x + k - [x+k]}{2008}$

$= [x] = \dfrac{1}{2008}\sum_{k=1}^{2008} x + k - ([x] + k)$

$= [x] + \dfrac{1}{2008}[2008(x - [x])]$

$\Rightarrow \ f(x) = [x] + x - [x] = x \quad \therefore \ f(3) = 3$

86. (9) : $\left[\sqrt{n+1}\right] \ne \left[\sqrt{n}\right]$ ...(1)

$0 < \sqrt{n+1} - \sqrt{n} < 1$

$\Rightarrow \left[\sqrt{n+1}\right] = \left[\sqrt{n}\right]$ or $\left[\sqrt{n}\right] + 1$ ...(2)

from (1) and (2), we get
$\left[\sqrt{n+1}\right] = \left[\sqrt{n}\right] + 1$

$k^2 \le n+1 < (k+1)^2 \Rightarrow [\sqrt{n+1}] = k$

$\Rightarrow \ n \ge k^2 - 1$ ...(3)
$\left[\sqrt{n}\right] = k - 1 \Rightarrow n < k^2$ ...(4)
from (3) and (4), $n = k^2 - 1$
$1 \le k^2 - 1 \le 100 \Rightarrow k = 2, 3, ..., 10$

87. (9) : $f(m) - f(n) = 3$
if $m = 125, n = 124 \Rightarrow (m-n)_{\min} = 1$
$m = 24, n = 5 \Rightarrow f(m) - f(n) = 3$
$\Rightarrow \ (m-n)_{\max} = 19.$
$\Rightarrow \ \dfrac{(m-n)_{\max} - (m-n)_{\min}}{2} = 9$

88. (6) : $x^2 + y^2 = 4$
Put $x = 2\cos\theta$, $y = 2\sin\theta$, we have

$$\left(\frac{x^3 + y^3}{x + y}\right) = x^2 + y^2 - xy = 4 - 2\sin 2\theta$$

$$\Rightarrow \left(\frac{x^3 + y^3}{x + y}\right)_{\max} = 4 - 2(-1) = 6$$

89. (1) : $f(2011) = (2 + 0 + 1 + 1)^2 = 16$
$f^2(2011) = (1 + 6)^2 = 49$
$f^3(2011) = (4 + 9)^2 = 169$
$f^4(2011) = (1 + 6 + 9)^2 = 256$
$f^5(2011) = (2 + 5 + 6)^2 = 169$
It is a periodic function.
$\Rightarrow f^{2n}(2011) = 256$
$f^{2n+1}(2011) = 169$

MOCK TEST PAPER 2015

JEE Main & Advanced Series-2

Complex Numbers : Algebra of complex numbers, addition, multiplication, conjugation, polar representation, properties of modulus and principal argument, triangle inequality, cube roots of unity, geometric interpretations.
Quadratic Equations : Quadratic equations with real coefficients, relation between roots and coefficients, formation of quadratic equations with given roots, symmetric functions of roots.
Trigonometry : General solution of trigonometric equations, properties of triangle-relation between sides and angles of a triangle, sine rule, cosine rule, half-angle formula and area of a triangle, inverse trigonometric functions (principal value only).
Co-ordinate geometry-2D : Equation of a straight line in various forms, angle between two lines, distance of a point from a line, lines through the point of intersection of two given lines, equation of the bisector of the angle between two lines, concurrency of lines, centroid, orthocentre, incentre and circumcentre of a triangle.

PAPER-I

Time : 2 hrs. Marks : 90

SECTION-I

Single Option Correct

[3 marks for correct answer and –1 for wrong answer]

This section contains 10 multiple choice questions. Each question has 4 choices, out of which ONLY ONE is correct.

1. The value of $\left\{\dfrac{1+\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}}{1+\cos\frac{\pi}{8}-i\sin\frac{\pi}{8}}\right\}^8 =$

(a) $1+i$ (b) $1-i$ (c) 1 (d) -1

2. If $\left|z+\dfrac{1}{z}\right|=3$, then the greatest value of $|z|$ is

(a) $3+\sqrt{3}$ (b) $3+\sqrt{13}$

(c) $\sqrt{13}-3$ (d) none of these

3. Let $3-i$ and $2+i$ be affixes of two points A and B in Argand plane and P represent the complex number $z = x + iy$ such that $|z-3+i| = |z-2-i|$. Then the locus of P is

(a) a circle on AB as diameter
(b) the line AB
(c) the perpendicular bisector of AB
(d) none of these

4. The quadratic equation $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ has one root in common. The other roots of the first and second equation are integers in the ratio 4 : 3. Then the common root is

(a) 2 (b) 1 (c) 4 (d) 3

5. If the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, then a, b, c are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

6. In a triangle with one angle $2\pi/3$, the length of the sides forms an A.P. If the length of the greatest side is 7 cm, then the radius of the circumcircle of the triangle is

(a) $\dfrac{7\sqrt{3}}{3}$ cm (b) $\dfrac{5\sqrt{3}}{3}$ cm

(c) $\dfrac{2\sqrt{3}}{3}$ cm (d) $7\sqrt{3}$ cm

7. The most general solution of

$2^{\sin x}+2^{\cos x}=2^{1-\frac{1}{\sqrt{2}}}$ are

By : Sankar Ghosh, HOD(Math), Takshyashila. Mob : 09831244397

(a) $n\pi-\frac{\pi}{4}$ (b) $n\pi+\frac{\pi}{4}$

(c) $n\pi+(-1)^n\frac{\pi}{4}$ (d) $2n\pi\pm\frac{\pi}{4}$

8. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y-intercept, is -4. Then a possible value of k is

(a) -4 (b) 1 (c) 2 (d) -2

9. Through the point (13, 31), a straight line is drawn to meet the axes of x and y at Q and S respectively. If the rectangle $OQRS$ is completed, then the locus $R(h, k)$ is

(a) $\frac{13}{x}+\frac{31}{y}=1$ (b) $\frac{31}{x}+\frac{13}{y}=1$

(c) $\frac{31}{x}-\frac{13}{y}=1$ (d) $\frac{13}{x}-\frac{31}{y}=1$

10. If the lines $ax+ky+10=0$, $bx+(k+1)y+10=0$ and $cx+(k+2)y+10=0$ are concurrent, then

(a) a, b, c are in G.P. (b) a, b, c are in H.P.

(c) a, b, c are in A.P. (d) $a+b=c$

SECTION-II

Multiple Correct Options

[4 marks for correct answer and –1 for wrong answer]

This section contains 5 multiple choice questions. Each question has 4 choices, out of which ONE or MORE is/are correct.

11. If in ΔABC, $a^4+b^4+c^4=2a^2(b^2+c^2)$, then $\angle A=$

(a) 45° (b) 60° (c) 90° (d) 135°

12. If $(1-\tan x)(1+\sin 2x)=1+\tan x$, then x is equal to

(a) $n\pi$ (b) $(2n+1)\pi$

(c) $n\pi-\frac{\pi}{4}$ (d) $2n\pi\pm\frac{\pi}{8}$

13. The equation $x^2-6x+8+\lambda(x^2-4x+3)=0$, $\lambda\in R$ has

(a) real and distinct roots $\forall\ \lambda$

(b) real roots $\forall\ \lambda<0$

(c) real roots $\forall\ \lambda>0$

(d) real and distinct roots for $\lambda=0$

14. The value of $\sum_{k=1}^{6}\left(\sin\frac{2k\pi}{7}-i\cos\frac{2k\pi}{7}\right)=$

(a) $-i$ (b) 0 (c) i (d) $-i^{-37}$

15. The lines $2x-y+1=0$, $(m+4)x-(2m-1)y=0$ and $4mx+(m-6)y+1=0$ are

(a) concurrent for two values of m

(b) concurrent for one value of m

(c) concurrent for no value of m

(d) parallel for $m=2$

SECTION-III

Assertion & Reason Type

[4 marks for correct answer and –1 for wrong answer]

This section contains 5 questions. Each question contains Statement-1 (Assertion) and Statement-2 (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

In each of the following questions two statements are given as Assertion (A) and Reason (R). Examine the statements carefully and answer the questions according to the instruction given below.

(a) if both A and R is correct and R is the proper reason of A.

(b) if both A and R is correct and R is not the proper reason of A.

(c) If A is correct and R is wrong.

(d) If A is wrong and R is correct.

16. **Statement 1 :** Consider the point $A(0, 1)$ and $B(2, 0)$ and P be a point on the line $4x+3y+9=0$, then the coordinates of P such that $|PA-PB|$ is maximum, is $\left(-\frac{84}{5}, \frac{13}{5}\right)$.

Statement 2 : If A and B are two fixed points and P is any point in a plane then $|PA-PB|\le AB$.

17. Let z be a moving point in a complex plane such that $\text{amp}\left(\frac{z-1}{z-2}\right)=\frac{\pi}{4}$.

Statement 1 : The locus of z will be a circle.

Statement 2 : If a point is moving such that its distance from the fixed point is always constant then the locus of the point is a circle.

18. **Statement 1 :** Incentre of a triangle formed by the lines $3x+4y=0$, $5x-12y=0$ and $y-15=0$ is the point P whose coordinates are (1, 8).

Statement 2 : Point P is equidistant from the 3 lines forming the triangle.

19. **Statement 1 :** If $a, b, c \in R$ and equations $ax^2 + bx + c = 0$ and $x^2 + 5x + 7 = 0$ has a common root, then $\frac{a+c}{b} = \frac{7}{5}$.

Statement 2 : If both roots of $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ are identical, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, where a_1, b_1, c_1 and $a_2, b_2, c_2 \in R$.

20. **Statement 1 :** If ΔABC is equilateral, then $\tan A + \tan B + \tan C = 3\sqrt{3}$.

Statement 2 : In ΔABC, $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

SECTION-IV

Integer Answer type

[4 marks for correct answer and –1 for wrong answer]

This section contains 5 questions. The answer to each of the questions is a single-digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like the following.

X	Y	Z	W
0	0	0	0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8
9	9	9	9

21. The smallest value of k for which both the roots of the equation $x^2 - 8xk + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is

22. Let $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex number z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \text{ is equal to}$$

23. Consider a ΔABC. Suppose $BC = 6$, $CA = 10$ and area of the triangle is $15\sqrt{3}$. If $\angle ACB > 90°$ and r denotes the radius of the incircle of the triangle, then r^2 is equal to

24. If z is a complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i| \le 2$ is

25. The number of values of x in $[0, 2\pi]$ such that $1 + \sin^4 2x = \cos^2 6x$, is

PAPER-II

Time : 2 hrs. Marks : 90

SECTION-I

Single Option Correct

This section contains 10 multiple choice questions numbered 1 to 4. Each question has 4 choices (a), (b), (c)and (d), out of which **ONLY ONE** is correct. [3 marks for correct answer and –1 for wrong answer]

1. If $|z| = 1$ and $z \ne \pm 1$, then one of the possible values of $\arg(z) - \arg(z + 1) - \arg(z - 1)$ is
(a) $-\pi/6$ (b) $\pi/3$ (c) $-\pi/2$ (d) $\pi/4$

2. Let c be a complex number such that $|c| = 1$. If the equation $cz^2 + z + 1 = 0$ has a purely imaginary root, then $\tan(\arg c) =$
(a) $\frac{\sqrt{5}-1}{2}$ (b) $\frac{\sqrt{5}+1}{2}$
(c) $\sqrt{\frac{\sqrt{5}-1}{2}}$ (d) $\sqrt{\frac{\sqrt{5}+1}{2}}$

3. If $|z - 4 + 3i| \le 1$ and α and β be the least and greatest value of $|z|$ and k be the least value of $\frac{x^4 + x^2 + 4}{x}$ on the interval $(0, \infty)$, then $k =$
(a) α (b) β
(c) $\alpha + \beta$ (d) the line AB

4. If the roots of quadratic equation $ax^2 + bx + c = 0$ are of the form $\frac{m}{m-1}$ and $\frac{m+1}{m}$, then the value of $(a + b + c)^2$ is
(a) $b^2 - 2ac$ (b) $2b^2 - ac$
(c) $b^2 - 4ac$ (d) $2(b^2 - 2ac)$

5. If two of the roots of the equation $(\lambda - 2)(x^2 + x + 1)^2 - (\lambda + 2)(x^4 + x^2 + 1) = 0$ are equal, then the value of λ is
(a) $\{-4, 4\}$ (b) $\{-2, 2\}$ (c) $\{2\}$ (d) $\{4\}$

6. If α, β and γ are the altitudes of the ΔABC from the vertices A, B and C respectively, then value of $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$ is

(a) $\dfrac{\cot A+\cot B+\cot C}{\Delta}$

(b) $\dfrac{\Delta}{\cot A+\cot B+\cot C}$

(c) $\dfrac{\cot A+\cot B+\cot C}{2\Delta}$

(d) $\dfrac{3}{\Delta}$

7. If in a triangle ABC, $(\cos A\cos B+\sin A\sin B)\sin^n C = 1$, $n \in N$, then the sides are in the ratio

(a) $1:1:\sqrt{2}$ (b) $\sqrt{2}:1:2$

(c) $\sqrt{2}:\sqrt{2}:1$ (d) none of these

8. The number of real solutions of (x, y), where $y = |\sin x|$, $y = \cos^{-1}(\cos x)$, $-2\pi \le x \le 2\pi$ is

(a) 2 (b) 1 (c) 3 (d) 4

9. The value of $\cot\left\{\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right\}$ is

(a) $\dfrac{23}{25}$ (b) $\dfrac{25}{23}$

(c) $\dfrac{23}{24}$ (d) $\dfrac{24}{23}$

10. The sides BC, CA, AB of ΔABC are respectively $x + 2y = 1$, $3x + y + 5 = 0$, $x - y + 2 = 0$. The altitude through B is

(a) $x - 3y + 1 = 0$ (b) $x - 3y + 4 = 0$

(c) $3x - y + 4 = 0$ (d) $x - y + 2 = 0$

SECTION-II

Multiple Correct type

This section contains 7 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which **ONE OR MORE** is/are correct. [4 marks for correct answer and –1 for wrong answer]

11. A point on the line $y = x$ whose perpendicular distance from the line $\frac{x}{4}+\frac{y}{3}=1$ is 4 has the coordinates

(a) $\left(-\frac{8}{7}, -\frac{8}{7}\right)$ (b) $\left(\frac{32}{7}, \frac{32}{7}\right)$

(c) $\left(\frac{3}{2}, \frac{3}{2}\right)$ (d) none of these

12. A ray of light travelling along the line $x + y = 1$ is incident on the x-axis and after refraction it enters the other side of the x-axis by turning $\pi/6$ away from the x-axis. The equation of the line along which the refracted ray travels is

(a) $x+(2-\sqrt{3})y=1$

(b) $(2-\sqrt{3})x+y=1$

(c) $y+(2+\sqrt{3})x=2+\sqrt{3}$

(d) none of these

13. If $0 \le x \le 2\pi$ and $|\cos x| \le \sin x$, then

(a) the set of x is $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

(b) the number of solutions that are integral multiples of $\pi/4$ is three

(c) $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$

(d) the sum of the largest and the smallest solution is $3\pi/4$.

14. If $f(x)=\cos^{-1}x+\cos^{-1}\left\{\frac{x}{2}+\frac{1}{2}\sqrt{3-3x^2}\right\}$, then

(a) $f\left(\frac{2}{3}\right)=\frac{\pi}{3}$

(b) $f\left(\frac{2}{3}\right)=2\cos^{-1}\frac{2}{3}-\frac{\pi}{3}$

(c) $f\left(\frac{1}{3}\right)=\frac{\pi}{3}$

(d) $f\left(\frac{1}{3}\right)=2\cos^{-1}\frac{1}{3}-\frac{\pi}{3}$

15. If α, β are real roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always

(a) two real roots

(b) two negative roots

(c) two positive roots

(d) one positive root and one negative root

SECTION-III

Linked Comprehension Type

This section contains 2 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which **ONLY ONE OR MORE** is/are correct. [Correct answer : 4 marks, wrong answer :–1 mark]

Paragraph for 16. to Q. 18

Consider the quadratic equation

$(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$,

where $m \in R - \{-1\}$.

16. The number of integral values of m such that given quadratic equation has imaginary roots are

(a) 0 (b) 1 (c) 2 (d) 3

17. The set of values of m such that the given quadratic equation has atleast one root is negative, is

(a) $m \in (-\infty, -1)$ (b) $m \in \left(-\frac{1}{8}, \infty\right)$

(c) $m \in \left(-1, -\frac{1}{8}\right)$ (d) $m \in R$

18. The number of real values of m such that the given quadratic equation has roots are in the ratio 2 : 3, are

(a) 0 (b) 2

(c) 4 (d) infinitely many

Paragraph for Q. 19 to Q. 21

Let S_1, S_2, S_3 be three sets of complex numbers such that $S_1 = \{z \in C : \text{Im}(z) \geq 1\}$, $S_2 = \{z \in C : |z - 2 - i| = 3\}$ and $S_3 = \{z \in C : \text{Re}((1 - i)z) = \sqrt{2}\}$

19. The number of elements in the set $S_1 \cap S_2 \cap S_3$ is

(a) 0 (b) 1 (c) 2 (d) ∞

20. Let z be any point in $S_1 \cap S_2 \cap S_3$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(a) 25 and 29 (b) 30 and 34

(c) 35 and 39 (d) 40 and 44

21. Let z be any point in $S_1 \cap S_2 \cap S_3$ and let ω be any point satisfying $|\omega - 2 - i| < 3$, then $|z| - |\omega| + 3$ lies between

(a) –6 and 3 (b) –3 and 6

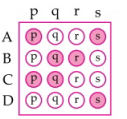
(c) –6 and 6 (d) –3 and 9

SECTION-IV

Matrix-Match Type

This section contains 3 questions.

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r and s. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example. If the correct matches are A-p and s ; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the above. [8 marks for correct answer and no negative marking for wrong answer and each row 2 marks]



22. Match the following:

	Column-I		Column-II
(A)	If in a ΔABC, $a = 6$, $b = 3$ and $\cos(A - B) = 4/5$, then area of ΔABC =	(p)	0
(B)	If $a = 5$, $b = 7$, $\sin A = 3/4$, then the number of triangles possible is/are	(q)	2
(C)	In a ΔABC, $b + c = 3a$, then $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) =$	(r)	3
(D)	$\frac{a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C}{abc} =$	(s)	9

23. Match the following:

	Column-I		Column-II
(A)	The value of $\tan^{-1}([\pi]) + \tan^{-1}([-\pi] + 1) =$	(p)	2
(B)	The number of solutions of the equation $\tan x + \sec x = 2\cos x$ in the interval $[0, 2\pi]$ is	(q)	$[\pi]$

(C)	The number of roots of the equation $x + 2\tan x = \pi/2$ in the interval $[0, 2\pi]$ is	(r)	0
(D)	The number of solutions of the equation $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \le x \le 2\pi$ is	(s)	1

24. Match the following:

Column-I		Column-II	
(A)	Values of θ for which the point $(\sin^2\theta, \sin\theta)$ lies on or inside the square formed by the lines $xy = 0$ and $4xy - 2x - 2y + 1 = 0$ is/are	(p)	3
(B)	$P(x_1, y_1)$ and $Q(x_2, y_2)$ satisfy the equation $\cot^2(x + y) + \tan^2(x + y) + y^2 + 2y - 1 = 0$, the minimum distance between P and Q is	(q)	$\pi/6$
(C)	The number of points (x, y) where $x, y \in N$, which lie inside the region bounded by $\lvert x\rvert + \lvert y\rvert = 5$ is/are	(r)	$\pi/4$
(D)	Area of the triangle formed by angle bisectors of coordinate axes and the line $x = \sqrt{6}$ is	(s)	6

SOLUTIONS

PAPER-I

1. (d) : $\left(\dfrac{1+\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}}{1+\cos\frac{\pi}{8}-i\sin\frac{\pi}{8}}\right)^8$

$= \left(\dfrac{2\cos^2\frac{\pi}{16}+2i\sin\frac{\pi}{16}\cos\frac{\pi}{16}}{2\cos^2\frac{\pi}{16}-2i\sin\frac{\pi}{16}\cos\frac{\pi}{16}}\right)^8$

$[\because \cos 2\theta = 2\cos^2\theta - 1, \sin 2\theta = 2\sin\theta\cos\theta]$

$= \left(\dfrac{\cos\frac{\pi}{16}+i\sin\frac{\pi}{16}}{\cos\frac{\pi}{16}-i\sin\frac{\pi}{16}}\right)^8 = \left(\dfrac{\left(\cos\frac{\pi}{16}+i\sin\frac{\pi}{16}\right)^2}{\cos^2\frac{\pi}{16}-i^2\sin^2\frac{\pi}{16}}\right)^8$

$= \left[\cos\dfrac{\pi}{8}+i\sin\dfrac{\pi}{8}\right]^8$

$[\because (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta]$

$= \cos\pi + i\sin\pi = -1$

Alternative method :

$\left(\dfrac{1+\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}}{1+\cos\frac{\pi}{8}-i\sin\frac{\pi}{8}}\right)^8 = \left(\cos\dfrac{\pi}{8}+i\sin\dfrac{\pi}{8}\right)^8$

$= \cos\pi + i\sin\pi = -1$

Remember :

(i) $\dfrac{1+\cos\theta+i\sin\theta}{1+\cos\theta-i\sin\theta} = \cos\theta + i\sin\theta$

(ii) $\dfrac{1+\cos\theta-i\sin\theta}{1+\cos\theta+i\sin\theta} = \cos\theta - i\sin\theta$

2. (d) : $|z| = \left|z + \dfrac{1}{z} - \dfrac{1}{z}\right|$

$\le \left|z + \dfrac{1}{z}\right| + \left|\dfrac{1}{z}\right|$ $\left[\because |z_1 - z_2| \le |z_1| + |z_2|\right]$

$\Rightarrow |z| \le 3 + \dfrac{1}{|z|} \Rightarrow |z|^2 - 3|z| - 1 \le 0$

For $|z|^2 - 3|z| - 1 = 0$,

$|z| = \dfrac{3 \pm \sqrt{9 - 4(1)(-1)}}{2 \cdot 1} = \dfrac{3 \pm \sqrt{13}}{2}$

$\therefore \left(|z| - \dfrac{3+\sqrt{13}}{2}\right) \le 0$

$\therefore$ The greatest value of $|z| = \dfrac{\sqrt{13}+3}{2}$

Alternative method :

The maximum value of $|z| = \dfrac{\sqrt{3^2+4}+3}{2}$

$= \dfrac{\sqrt{13}+3}{2}$

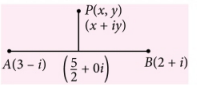
Remember : If $\left|z+\frac{1}{z}\right|=a$, then

(i) maximum value of $|z| = \frac{\sqrt{a^2+4}+a}{2}$

(ii) minimum value of $|z| = \frac{\sqrt{a^2+4}-a}{2}$

where a is a positive real number.

3. (c):



Given that, $|z-3+i| = |z-2-i|$

$\Rightarrow |(x-3)+i(y+1)| = |(x-2)+i(y-1)|$

$\Rightarrow (x-3)^2+(y+1)^2 = (x-2)^2+(y-1)^2$

$\Rightarrow 2x-4y=5$ ... (i)

The line (i) passes through the point $\left(\frac{5}{2}, 0\right)$ which is the mid-point of AB.

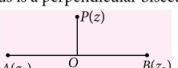
Again, slope of $AB = m_1 = \frac{1+1}{2-3} = -2$

and slope of the line (i), $m_2 = \frac{1}{2}$

$\therefore\ m_1 \times m_2 = -1$

Thus, the locus is a perpendicular bisector of AB.

Remember :



Let $P(z)$ be any point on the perpendicular bisector of AB.

Then $PA = PB \Rightarrow |z-z_1| = |z-z_2|$

which is the required equation of the perpendicular bisector of segment joining $A(z_1)$ and $B(z_2)$.

4. (a) : Let α and 4β be the roots of the equation $x^2-6x+a=0$

and α and 3β be those of the equation $x^2-cx+6=0$

$\therefore\ \alpha+4\beta=6$ and $4\alpha\beta=a$

$\alpha+3\beta=c$ and $3\alpha\beta=6$

We get, $\alpha\beta=2$ and $a=8$

The first equation becomes $x^2-6x+8=0$

$\Rightarrow\ x=2, 4$

For $\alpha=2$, $4\beta=4 \Rightarrow 3\beta=3$

For $\alpha=4$, $4\beta=2 \Rightarrow 3\beta=3/2$ (not an integer)

So, the common root is $\alpha=2$.

5. (c): The given equation is

$a(b-c)x^2+b(c-a)x+c(a-b)=0$ ... (i)

Let $A=a(b-c)$, $B=b(c-a)$ and $C=c(a-b)$

$\therefore\ A+B+C=0$

Now, (i) becomes $Ax^2+Bx+C=0$

$\because$ roots are equal $\therefore\ B^2=4AC$

$\Rightarrow (A+C)^2-4AC=0$ $[\because A+B+C=0]$

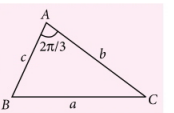
$\Rightarrow (A-C)^2=0 \Rightarrow A=C$

$\Rightarrow a(b-c)=c(a-b) \Rightarrow ab-ac=ac-bc$

$\Rightarrow ab+bc=2ac \Rightarrow \frac{1}{a}+\frac{1}{c}=\frac{2}{b}$

$\therefore$ a, b, c are in H.P.

6. (a) :



$\therefore\ \frac{7}{\sin\frac{2\pi}{3}}=2R$ (Using sine formula)

$\Rightarrow \frac{7\times2}{\sqrt{3}}=2R \Rightarrow R=\frac{7}{\sqrt{3}}=\frac{7\sqrt{3}}{3}$ cm

7. (b) : We know, A.M. $\geq$ G.M.

$\Rightarrow \frac{2^{\sin x}+2^{\cos x}}{2} \geq \sqrt{2^{\sin x}\cdot 2^{\cos x}}$

$\Rightarrow 2^{\sin x}+2^{\cos x} \geq 2\cdot 2^{\frac{\sin x+\cos x}{2}} \geq 2^{1-\frac{1}{\sqrt{2}}}$

$$\left[\begin{array}{l}\sin x+\cos x=\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)\\ \qquad\qquad =\sqrt{2}\cos\left(\frac{\pi}{4}-x\right)\\ \therefore \text{ minimum value of } \sin x+\cos x=-\sqrt{2}\end{array}\right]$$

The equation holds when equality holds

i.e. $2^{\sin x}=2^{\cos x} \Rightarrow \sin x=\cos x$

$\Rightarrow \tan x=1=\tan\frac{\pi}{4} \Rightarrow x=n\pi+\frac{\pi}{4}$

8. (a) : The mid-point of PQ is

$$\left(\frac{1+k}{2}, \frac{4+3}{2}\right)=\left(\frac{1+k}{2}, \frac{7}{2}\right)$$

Slope of $PQ = \dfrac{3-4}{k-1} = -\dfrac{1}{k-1}$

Let the slope of the perpendicular bisector of PQ be m

$\therefore\ m\left(-\dfrac{1}{k-1}\right) = -1 \ \Rightarrow\ m = k-1$

Equation of bisector of PQ is

$$y - \frac{7}{2} = (k-1)\left(x - \frac{1+k}{2}\right)$$

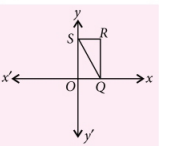
$$\Rightarrow\ y = (k-1)x - \frac{k^2-1}{2} + \frac{7}{2}$$

$$\Rightarrow\ (1-k)x + y = \frac{7-k^2+1}{2} = \frac{8-k^2}{2}$$

$$\Rightarrow\ \frac{x}{\dfrac{8-k^2}{2(1-k)}} + \frac{y}{\dfrac{8-k^2}{2}} = 1$$

$$\therefore\ \frac{8-k^2}{2} = -4 \ \Rightarrow\ k^2 = 16 \ \Rightarrow\ k = \pm 4$$

9. (a) : Equation of the line passing through the point (13, 31) is



$$y - 31 = mx - 13m$$
$$= m(x - 13)$$

$$\Rightarrow\ mx - y = 13m - 31$$

$$\Rightarrow\ \frac{x}{\dfrac{13m-31}{m}} + \frac{y}{-(13m-31)} = 1$$

$$\therefore\ Q \equiv \left(\frac{13m-31}{m}, 0\right) \text{ and } S \equiv (0, 31 - 13m)$$

As rectangle $OQRS$ is completed

$$\therefore\ R \equiv \left(13 - \frac{31}{m},\ 31 - 13m\right) = (h, k) \text{ (say)}$$

Eliminating m, we get

$$h = 13 - \frac{31}{m} \quad \text{and} \quad k = 31 - 13m$$

$$\Rightarrow\ \frac{31}{m} = 13 - h \ \text{ and } \ 13m = 31 - k$$

$$\Rightarrow\ m = \frac{31}{13-h} \ \text{ and } \ m = \frac{31-k}{13}$$

$$\therefore\ \frac{31}{13-h} = \frac{31-k}{13} \ \Rightarrow\ 31h + 13k = hk$$

$$\Rightarrow\ \frac{31}{y} + \frac{13}{x} = 1, \text{ which is the required locus.}$$

10. (c): The given lines are

$$ax + ky + 10 = 0 \qquad \ldots (i)$$
$$bx + (k+1)y + 10 = 0 \qquad \ldots (ii)$$
$$cx + (k+2)y + 10 = 0 \qquad \ldots (iii)$$

Solving (i) and (ii), we get

$$x = \frac{-10}{a(k+1) - bk},\ y = \frac{10(b-a)}{a(k+1) - bk}$$

Since, (i), (ii) and (iii) are concurrent

$$\therefore\ \frac{-10c}{a(k+1) - bk} + \frac{10(k+2)(b-a)}{a(k+1) - bk} + 10 = 0$$

$$\Rightarrow\ -c + (k+2)(b-a) + a(k+1) - bk = 0$$
$$\Rightarrow\ -c + 2b - a = 0 \ \Rightarrow\ a + c = 2b$$
$$\therefore\ a, b, c \in \text{A.P.}$$

11. (a, d) : Given that in any ΔABC,

$$a^4 + b^4 + c^4 = 2a^2(b^2 + c^2)$$
$$\Rightarrow\ a^4 + b^4 + c^4 = 2a^2b^2 + 2a^2c^2$$
$$\Rightarrow\ (a^2)^2 + (b^2)^2 + (c^2)^2 - 2a^2b^2 - 2a^2c^2 = 0$$
$$\Rightarrow\ (b^2)^2 + (c^2)^2 + (-a^2)^2 + 2b^2c^2 - 2a^2b^2 - 2a^2c^2 = 2b^2c^2$$
$$\Rightarrow\ (b^2 + c^2 - a^2)^2 = 2b^2c^2$$
$$\Rightarrow\ b^2 + c^2 - a^2 = \pm\sqrt{2}bc \ \Rightarrow\ \frac{b^2 + c^2 - a^2}{2bc} = \pm\frac{1}{\sqrt{2}}$$
$$\therefore\ \cos A = \pm\frac{1}{\sqrt{2}} \ \ \therefore\ \angle A = 45^\circ \text{ or } 135^\circ$$

12. (a, c): $(1 - \tan x)(1 + \sin 2x) = 1 + \tan x$

$$\Rightarrow\ \frac{(\cos x - \sin x)(\cos x + \sin x)^2}{\cos x} = 1 + \tan x$$
$$\Rightarrow\ (\cos^2 x - \sin^2 x)(1 + \tan x) = (1 + \tan x)$$
$$\Rightarrow\ (1 + \tan x)(\cos 2x - 1) = 0$$
$$\Rightarrow\ \cos 2x = 1 \ \Rightarrow\ 2x = 2n\pi \ \Rightarrow\ x = n\pi$$

and $1 + \tan x = 0 \ \Rightarrow\ \tan x = -1 = \tan\left(-\dfrac{\pi}{4}\right)$

$$\Rightarrow\ x = n\pi - \frac{\pi}{4}$$
$$\therefore\ x = n\pi,\ n\pi - \frac{\pi}{4}$$

13. (a, b, c, d) : The given equation is

$$x^2 - 6x + 8 + \lambda(x^2 - 4x + 3) = 0,\ \lambda \in R$$
$$\Rightarrow\ (1 + \lambda)x^2 - 2(3 + 2\lambda)x + 8 + 3\lambda = 0$$

Now, $\{-2(3 + 2\lambda)\}^2 - 4(1 + \lambda)(8 + 3\lambda)$

$$= 4(3 + 2\lambda)^2 - 4(1 + \lambda)(8 + 3\lambda)$$
$$= 4[9 + 12\lambda + 4\lambda^2 - 8 - 11\lambda - 3\lambda^2]$$
$$= 4[\lambda^2 + \lambda + 1]$$
$$= 4\left[(\lambda)^2 + 2\lambda \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 + 1 - \frac{1}{4}\right]$$
$$= 4\left[\left(\lambda + \frac{1}{2}\right)^2 + \frac{3}{4}\right] > 0 \ \forall \lambda \in R$$

$\Rightarrow$ roots of the equation are real and distinct.

14. (c, d) : $\sum_{k=1}^{6}\left(\sin\frac{2k\pi}{7} - i\cos\frac{2k\pi}{7}\right) = i$

and $-i^{-37} = -\frac{1}{i^{37}} = -\frac{1}{(i^2)^{18} \cdot i} = -\frac{1}{i} = \frac{i^2}{i} = i$

Remember : $\sum_{k=1}^{n}\left(\sin\frac{2k\pi}{n+1} - i\cos\frac{2k\pi}{n+1}\right) = i$

and $\sum_{k=1}^{n}\left(\cos\frac{2k\pi}{n+1} + i\sin\frac{2k\pi}{n+1}\right) = -1$

15. (a, d) : $2x - y + 1 = 0$ … (i)

$(m + 4)x - (2m - 1)y + 0 = 0$ … (ii)

$4mx + (m - 6)y + 1 = 0$ … (iii)

Solving (i) and (ii), we get

$$x = \frac{2m-1}{(m+4)(m-6) + 4m(2m-1)} = \frac{2m-1}{9m^2 - 6m + 24}$$

$$y = \frac{m+4}{(m+4)(m-6) + 4m(2m-1)} = \frac{m+4}{9m^2 - 6m + 24}$$

Since the above three lines are concurrent,

$$\therefore \ \frac{4m(2m-1)}{9m^2 - 6m + 24} + \frac{(m-6)(m+4)}{9m^2 - 6m + 24} + 1 = 0$$

$\Rightarrow 8m^2 - 4m + m^2 - 2m - 24 + 9m^2 - 6m + 24 = 0$

$\Rightarrow 18m^2 - 12m = 0 \ \Rightarrow \ 6m(3m - 2) = 0$

$\Rightarrow m = 0$ or $m = 2/3$

$\therefore$ Equation (ii) and (iii) for $m = 2$ becomes

$6x - 3y = 0$ and $8x - 4y + 1 = 0$

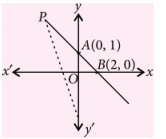
i.e. $2x - y = 0$ and $2x - y + 1 = 0$

Thus for $m = 2$, the lines are parallel.

16. (a) : $|PA - PB| \le AB$, $|PA - PB| = AB$

Thus, $|PA - PB|$ to be maximum, A, B and P must be collinear.

Equation of AB is $\frac{x}{2} + \frac{y}{1} = 1$



$\Rightarrow x + 2y = 2$ … (i)

Now, solving (i) with the given line $4x + 3y + 9 = 0$, we get, $P \equiv \left(-\frac{84}{5}, \frac{13}{5}\right)$

17. (c): Given, $\text{amp}\left(\frac{z-1}{z-2}\right) = \frac{\pi}{4}$

Let $z = x + iy$ $\therefore \ \text{amp}\left(\frac{(x-1)+iy}{(x-2)+iy}\right) = \frac{\pi}{4}$

or $\tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x-2}\right) = \frac{\pi}{4}$

$$\left[\because \text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2\right]$$

$$\Rightarrow \ \tan^{-1}\left\{\frac{\frac{y}{x-1} - \frac{y}{x-2}}{\frac{(x-1)(x-2) + y^2}{(x-1)(y-2)}}\right\} = \frac{\pi}{4}$$

$$\Rightarrow \ \frac{y(x-2) - y(x-1)}{(x-1)(x-2) + y^2} = 1$$

$\Rightarrow x^2 + y^2 - 3x + y + 2 = 0$

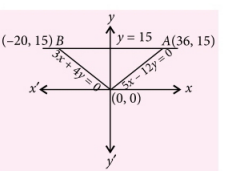
which is the equation of a circle.

18. (c): The given three lines are

$y = 15$ … (i)

$5x - 12y = 0$ … (ii)

$3x + 4y = 0$ … (iii)



From the diagram, it is clear that the vertices are $O(0, 0)$, $A(36, 15)$ and $B(-20, 15)$

Now, $AB = 56$

$OA = \sqrt{36^2 + 15^2} = 39$

$OB = \sqrt{20^2 + 15^2} = 25$

$\therefore$ Incentre $= \left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$

$= \left(\dfrac{56\cdot 0+39(-20)+25\times 36}{56+39+25}, \dfrac{56\cdot 0+39(15)+25\times 15}{56+39+25}\right)$

$= (1, 8)$

Also P is equidistant from 3 lines forming triangle.

19. (d) : Let the common root be α

$\therefore\ a\alpha^2 + b\alpha + c = 0$ and $\alpha^2 + 5\alpha + 7 = 0$

$\Rightarrow\ \dfrac{\alpha^2}{7b-5c} = \dfrac{\alpha}{c-7a} = \dfrac{1}{5a-b}$

$\therefore\ (c-7a)^2 = (7b-5c)(5a-b)$

$\Rightarrow\ 49a^2 + 7b^2 + c^2 + 6ac - 35ab = 0$

Thus the statement-1 is false and statement-2 is true.

20. (c): In an equilateral triangle, $A = B = C = \dfrac{\pi}{3}$

$\Rightarrow\ \tan A = \tan B = \tan C = \sqrt{3}$

$\therefore\ \tan A + \tan B + \tan C = 3\sqrt{3}$

Again, in ΔABC, $A + B = \pi - C$

$\therefore\ \tan(A+B) = \tan(\pi - C)$

$\Rightarrow\ \tan A + \tan B = -\tan C(1 - \tan A \tan B)$

$\Rightarrow\ \tan A + \tan B + \tan C = \tan A \tan B \tan C$

21. (2) : The given equation is

$x^2 - 8kx + 16(k^2 - k + 1) = 0$

For real and distinct roots,

$64k^2 - 4\cdot 1\cdot 16(k^2 - k + 1) > 0$

$\Rightarrow\ k > 1$

Again, $\alpha \geq 4, \beta \geq 4\ \Rightarrow\ \alpha - 4 \geq 0, (\beta - 4) \geq 0$

$\therefore\ (\alpha - 4) + (\beta - 4) \geq 0\ \Rightarrow\ \alpha + \beta \geq 8$

$\Rightarrow\ 8k \geq 8\ \Rightarrow\ k \geq 1$

Also, $(\alpha - 4)(\beta - 4) \geq 0$

$\Rightarrow\ \alpha\beta - 4(\alpha + \beta) + 16 \geq 0$

$\Rightarrow\ 16(k^2 - k + 1) - 4\cdot 8k + 16 \geq 0$

$\Rightarrow\ k^2 - k + 1 - 2k + 1 \geq 0$

$\Rightarrow\ k^2 - 3k + 2 \geq 0\ \Rightarrow\ (k-1)(k-2) \geq 0$

$\Rightarrow\ k \geq 2$

$\therefore$ The smallest value of k is 2.

22. (1) : $\omega = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$

Now, $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$

Applying $C_1' = C_1 + C_2 + C_3$, we get

$\Rightarrow\ \begin{vmatrix} z & \omega & \omega^2 \\ z & z+\omega^2 & 1 \\ z & 1 & z+\omega \end{vmatrix} = 0 \quad (\because\ 1 + \omega + \omega^2 = 0)$

$\Rightarrow\ z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$

Applying $R_1' = R_1 - R_2$ and $R_2' = R_2 - R_3$, we get

$\Rightarrow\ z\begin{vmatrix} 0 & \omega - z - \omega^2 & \omega^2 - 1 \\ 0 & z+\omega^2 - 1 & 1 - z - \omega \\ 1 & 1 & z+\omega \end{vmatrix} = 0$

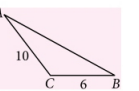
$\Rightarrow\ z\{(\omega - z - \omega^2)(1 - z - \omega) - (z + \omega^2 - 1)(\omega^2 - 1)\} = 0$

$\Rightarrow\ z(z^2 + \omega - \omega) = 0\ \Rightarrow\ z^3 = 0$

$\Rightarrow$ The number of distinct complex number z is 1.

23. (3) : $\sin C = \dfrac{2\Delta}{ab} = \dfrac{2\times 15\times\sqrt{3}}{6\times 10}$

$\sin C = \dfrac{\sqrt{3}}{2} = \sin 120°$



$\therefore\ C = 120°$

But $c^2 = a^2 + b^2 - 2ab\cos C$

$= 36 + 100 - 2\times 6\times 10\left(-\dfrac{1}{2}\right)$

$\Rightarrow\ c^2 = 196\ \ \therefore\ \ c = 14$

$r = \dfrac{\Delta}{s} = \dfrac{15\sqrt{3}}{\frac{10+6+14}{2}} = \dfrac{15\sqrt{3}}{15} = \sqrt{3}\ \ \therefore\ \ r^2 = 3$

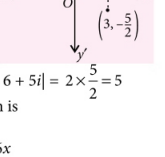
24. (5) : $|2z - 6 + 5i| = 2\left|z - \left(3 - \dfrac{5}{2}i\right)\right|$

$= 2\ \times$ distance between z and the point $\left(3, -\dfrac{5}{2}\right)$

But $|z - 3 - 2i| \leq 2$

$\Rightarrow\ |z - (3 + 2i)| \leq 2$

$\Rightarrow$ z is a point on or inside a circle whose radius is 2 and the centre is (3, 2).



$\therefore$ Minimum value of $|2z - 6 + 5i| = 2\times\dfrac{5}{2} = 5$

25. (5) : The given equation is

$1 + \sin^4 2x = \cos^2 6x$

$\Rightarrow\ 1 + (1 - \cos^2 2x)^2 = \cos^2 6x$

$\Rightarrow\ 1 + (1 - \cos^2 2x)^2 = (4\cos^3 2x - 3\cos 2x)^2$

Let $\cos 2x = a$

$\therefore \quad 1 + (1 - a^2)^2 = (4a^3 - 3a)^2$

$\Rightarrow \quad 16a^6 - 25a^4 + 11a^2 - 2 = 0$

The real roots are $a = \pm 1 \quad \Rightarrow \cos 2x = \pm 1$

$\Rightarrow \quad x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$

PAPER-II

1. (c): $\text{Arg}\,z - \text{Arg}(z+1) - \text{Arg}(z-1)$

$= \text{Arg}\left(\dfrac{z}{z^2-1}\right)$

$= \text{Arg}\left(\dfrac{z}{z^2 - z\bar{z}}\right)$ [$\because \ |z| = 1$ and $z\bar{z} = |z|^2$]

$= \text{Arg}\left(\dfrac{1}{z-\bar{z}}\right)$

$= \text{Arg}\left\{\dfrac{1}{(x+iy)-(x-iy)}\right\}$ [Let $z = x + iy$]

$= \text{Arg}\left(\dfrac{1}{2iy}\right) = \text{Arg(pure imaginary no.)} = -\pi/2$

2. (d) : Let $z = iy, y \in R$ and $c = \cos\alpha + i\sin\alpha$

$\therefore \quad cz^2 + z + 1 = 0$ gives

$-(\cos\alpha + i\sin\alpha)y^2 + iy + 1 = 0$

$\Rightarrow \quad (\cos\alpha + i\sin\alpha)y^2 - iy - 1 = 0$

$\Rightarrow \quad (\cos\alpha \cdot y^2 - 1) + (\sin\alpha \cdot y^2 - y)i = 0$

$\Rightarrow \quad y^2\cos\alpha - 1 = 0$ and $y^2\sin\alpha - y = 0$

$\Rightarrow \quad y^2\cos\alpha = 1, \ y\sin\alpha = 1, \ y \neq 0$

$\Rightarrow \quad y^2\cos\alpha - y^2\sin^2\alpha = 0 \quad \Rightarrow \quad y^2(\sin^2\alpha - \cos\alpha) = 0$

$\therefore \quad 1 - \cos^2\alpha - \cos\alpha = 0$

$\Rightarrow \quad \cos^2\alpha + \cos\alpha - 1 = 0$

$\Rightarrow \quad \cos\alpha = \dfrac{-1 \pm \sqrt{1^2 - 4 \cdot 1(-1)}}{2 \cdot 1} = \dfrac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \quad \cos\alpha = \dfrac{\sqrt{5}-1}{2} \quad \Rightarrow \quad \sec^2\alpha = \left(\dfrac{\sqrt{5}+1}{2}\right)^2$

$\therefore \quad \tan^2\alpha = \left(\dfrac{\sqrt{5}+1}{2}\right)^2 - 1 = \dfrac{\sqrt{5}+1}{2}$

$\therefore \quad \tan\alpha = \tan(\arg c) = \sqrt{\dfrac{\sqrt{5}+1}{2}}$

3. (b) : Given that $|z - 4 + 3i| \leq 1$

$\Rightarrow \quad |z - (4 - 3i)| \leq 1$

i.e. $1 \geq |z - (4 - 3i)|$

$\geq \begin{cases} |z| - |4 - 3i| \\ |4 - 3i| - |z| \end{cases}$

$\therefore \quad |z| \leq 6$ and $|z| \geq 4 \quad [\because \ |4 - 3i| = 5]$

$\Rightarrow \quad 4 \leq |z| \leq 6$

i.e. $\alpha = 4$ and $\beta = 6$

$\therefore \quad y$ will be least when $x^3 = x = \dfrac{1}{x} \quad \Rightarrow \quad x = 1$

as $y = \dfrac{x^4 + x^2 + 4}{x} = x^3 + x + \dfrac{4}{x}$

$\therefore \quad y_{(\text{least})} = 6. \quad \therefore \quad k = 6, \ \text{so} \ k = \beta$

4. (c): The roots of the equation

$ax^2 + bx + c = 0$ are $\dfrac{m}{m-1}$ and $\dfrac{m+1}{m}$

$\therefore \quad \dfrac{m+1}{m} + \dfrac{m}{m-1} = -\dfrac{b}{a}$ … (i)

and $\dfrac{m+1}{m} \cdot \dfrac{m}{m-1} = \dfrac{m+1}{m-1} = \dfrac{c}{a}$ … (ii)

From (ii), we get $m = -\left(\dfrac{a+c}{a-c}\right)$

Now, (i) becomes $\dfrac{2m^2 - 1}{m^2 - m} = -\dfrac{b}{a}$

$\Rightarrow \quad \dfrac{2\left(\dfrac{a+c}{a-c}\right)^2 - 1}{\left(\dfrac{a+c}{a-c}\right)^2 + \left(\dfrac{a+c}{a-c}\right)} = -\dfrac{b}{a}$

$\Rightarrow \quad \dfrac{2(a+c)^2 - (a-c)^2}{(a+c)^2 + a^2 - c^2} = -\dfrac{b}{a}$

$\Rightarrow \quad \dfrac{(a+c)^2 + 4ac}{2a^2 + 2ac} = -\dfrac{b}{a}$

$\Rightarrow \quad (a + c)^2 + 4ac = -2ab - 2bc$

$\Rightarrow \quad (a + c)^2 = -2ab - 2bc - 4ac$

$\Rightarrow \quad (a + c)^2 + 2(a + c)b + b^2$

$= 2(a + c)b + b^2 - 2ab - 2bc - 4ac$

$\Rightarrow \quad (a + c + b)^2 = b^2 - 4ac$

$\Rightarrow \quad (a + b + c)^2 = b^2 - 4ac$

5. (a): $(\lambda - 2)(x^2 + x + 1)^2 - (\lambda + 2)(x^4 + x^2 + 1) = 0$

$\Rightarrow \quad (\lambda - 2)(x^2 + x + 1)^2 - (\lambda + 2)(x^2 + x + 1)$

$(x^2 - x + 1) = 0$

$\Rightarrow (x^2+x+1)\{(x^2+x+1)(\lambda-2) - (\lambda+2)(x^2-x+1)\} = 0$

$\Rightarrow (x^2+x+1)[\{(\lambda-2)-(\lambda+2)\}x^2 + \{(\lambda-2) + (\lambda+2)\}x + (\lambda-2)-(\lambda+2)] = 0$

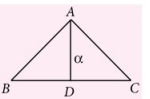
$\Rightarrow (x^2+x+1)(-4x^2+2\lambda x-4) = 0$

$\because \quad 2x^2 - \lambda x + 2 = 0$

$\therefore \quad \lambda^2 - 4\cdot 2\cdot 2 = 0 \Rightarrow \lambda = \pm 4$

6. (a) : Clearly, $\Delta = \frac{1}{2}\cdot a\cdot\alpha$

$\Rightarrow \frac{1}{\alpha} = \frac{a}{2\Delta}$



Similarly $\frac{1}{\beta} = \frac{b}{2\Delta}$

and $\frac{1}{\gamma} = \frac{c}{2\Delta}$

$\therefore \quad \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{a^2+b^2+c^2}{4\Delta^2} = \frac{1}{\Delta}\cdot\frac{a^2+b^2+c^2}{4\Delta}$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{1}{\Delta}(\cot A+\cot B+\cot C)$$

[Note that : $\Delta = \frac{abc}{4R}$ and

$$\cot A+\cot B+\cot C = \frac{R}{abc}(a^2+b^2+c^2)$$

$$= \frac{R(a^2+b^2+c^2)}{4R\Delta} = \frac{a^2+b^2+c^2}{4\Delta}]$$

7. (a) : We know, $0 \le \sin C \le 1 \Rightarrow 0 \le \sin^n C \le 1$

$\therefore \quad (\cos A\cos B + \sin A\sin B)\sin^n C \le \cos(A-B)$

Equality holds when $C = \pi/2$

From question, $1 \le \cos(A-B)$

But $\cos(A-B) \le 1$

$\therefore$ We get, $\cos(A-B) = 1$ and $C = \pi/2$

$\Rightarrow A = B$ and $C = \pi/2$

$\therefore \quad A = \frac{\pi}{4} - B$ and $C = \frac{\pi}{2}$

We know that $\frac{a}{\sin\frac{\pi}{4}} = \frac{b}{\sin\frac{\pi}{4}} = \frac{c}{\sin\frac{\pi}{2}}$

Thus, the ratio of the sides is $1:1:\sqrt{2}$.

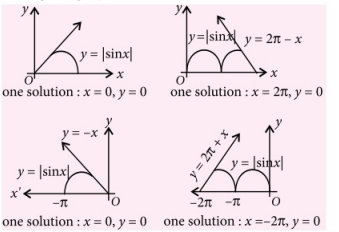
8. (c): (i) $\left.\begin{aligned} y &= |\sin x| = \sin x \\ y &= \cos^{-1}(\cos x) = x \end{aligned}\right\}$ when $x \in [0, \pi]$

(ii) $\left.\begin{aligned} y &= |\sin x| = -\sin x \\ y &= \cos^{-1}(\cos x) \\ &= \cos^{-1}\{\cos(2\pi - x)\} \\ &= 2\pi - x \end{aligned}\right\}$ when $x \in [\pi, 2\pi]$

(iii) $y = |\sin x|$, $y = \cos^{-1}(\cos(-x)) = -x$ when $x \in [-\pi, 0]$

(iv) $\left.\begin{aligned} y &= |\sin x| \\ y &= \cos^{-1}\{\cos(2\pi + x)\} \\ &= 2\pi + x \end{aligned}\right\}$ when $x \in [-2\pi, -\pi]$

Plotting the graphs,



There are three solutions (0, 0), (2π, 0) and (−2π, 0).

9. (b) : $\cot\left\{\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right\}$

$= \cot\left\{\sum_{n=1}^{23}\cot^{-1}(1+n(n+1))\right\}$

$= \cot\left\{\sum_{n=1}^{23}\tan^{-1}\left(\frac{1}{1+n(n+1)}\right)\right\}$

$= \cot\left\{\sum_{n=1}^{23}\tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right)\right\}$

$= \cot\left\{\sum_{n=1}^{23}(\tan^{-1}(n+1)-\tan^{-1}n)\right\}$

$= \cot\{(\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + (\tan^{-1}4 - \tan^{-1}3) + ... + (\tan^{-1}24 - \tan^{-1}23)\}$

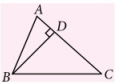
$= \cot(\tan^{-1}24 - \tan^{-1}1)$

$= \cot\left[\tan^{-1}\left(\frac{24-1}{1+24}\right)\right] = \cot\left[\tan^{-1}\left(\frac{23}{25}\right)\right]$

$$= \cot\left(\cot^{-1}\left(\frac{25}{23}\right)\right) = \frac{25}{23}$$

10. (b) : Equation of BC is

$x + 2y = 1$ … (i)



Equation of CA is

$3x + y + 5 = 0$ … (ii)

and equation of AB is $x - y + 2 = 0$ … (iii)

Solving (i) and (iii), we get $(-1, 1)$ is the vertex of B

Let the equation of altitude BD be

$x - 3y + k = 0$ … (iv)

Since (iv) is passing through $B(-1, 1)$

$-1 - 3 + k = 0 \Rightarrow k = 4$

$\therefore$ (iv) becomes $x - 3y + 4 = 0$

which is the required equation.

11. (a, b): Let the point on the line $y = x$ be (k, k)

Equation of the given line is

$$\frac{x}{4} + \frac{y}{3} = 1 \quad i.e. \quad 3x + 4y = 12$$

By the condition, we have

$$\frac{|3k + 4k - 12|}{\sqrt{3^2 + 4^2}} = 4 \Rightarrow |7k - 12| = 20$$

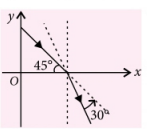
$\Rightarrow 7k - 12 = \pm 20 \Rightarrow 7k = 32$ or $7k = -8$

$\therefore k = \frac{32}{7}$ or $k = \frac{-8}{7}$

$\therefore$ The required points are

$$\left(\frac{32}{7}, \frac{32}{7}\right) \text{ and } \left(\frac{-8}{7}, \frac{-8}{7}\right)$$

12. (a, c):



Clearly the line of refracted ray passes through $(1, 0)$ and it has slope 105°

$\therefore$ The equation of the line of refracted ray is

$y - 0 = \tan 105°(x - 1)$

$\Rightarrow y = -(2 + \sqrt{3})x + (2 + \sqrt{3})$

$\Rightarrow (2 + \sqrt{3})x + y = 2 + \sqrt{3}$

$\Rightarrow x + (2 - \sqrt{3})y = 1$

13. (b, c): $|\cos x| \geq 0$. Given, $|\cos x| \leq \sin x \Rightarrow \sin x \geq 0$

So, there is no x in $(0, 2\pi)$

Now of $x = 2\pi$, $|\cos 2\pi| \leq \sin 2\pi$, which is not true.

So $0 \leq x \leq \pi$

If $0 \leq x \leq \frac{\pi}{2}$ then $|\cos x| \leq \sin x \Rightarrow \cos x \leq \sin x$

$\Rightarrow \tan x \geq 1 \quad \therefore \frac{\pi}{4} \leq x \leq \frac{\pi}{2}$ … (i)

If $\frac{\pi}{2} < x \leq \pi$ then $|\cos x| \leq \sin x$

$\Rightarrow -\cos x \leq \sin x$

$\Rightarrow \tan x \leq -1 \quad (\because \cos x < 0)$

$\therefore x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right]$ ... (ii)

From (i) and (ii), we can say that the number of solutions that are integral multiples of $\pi/4$ is three,

i.e. $\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$

14. (a,d): $f(x) = \cos^{-1} x + \cos^{-1}\left\{\frac{1}{2}x + \frac{\sqrt{3}}{2}\sqrt{1 - x^2}\right\}$

$= \cos^{-1} x \pm \left(\cos^{-1}\frac{1}{2} - \cos^{-1} x\right)$

$\left[\text{according as } \cos^{-1}\frac{1}{2} > \text{or} < \cos^{-1} x\right]$

$= \cos^{-1}\frac{1}{2}$

$\left[\text{When } \cos^{-1}\frac{1}{2} > \cos^{-1} x, \text{ which holds for } x = \frac{2}{3}\right]$

$= 2\cos^{-1} x - \cos^{-1}\frac{1}{2}$

$\left[\text{When } \cos^{-1}\frac{1}{2} < \cos^{-1} x, \text{ which holds for } x = \frac{1}{3}\right]$

15. (a, d) : α, β are the roots of $x^2 + px + q = 0$

$\therefore \alpha + \beta = -p$ and $\alpha\beta = q$

Again, α^4, β^4 are the roots of $x^2 - rx + s = 0$

$\therefore \alpha^4 + \beta^4 = r$ and $(\alpha\beta)^4 = s$

Now, discriminant of $x^2 - 4qx + 2q^2 - r = 0$ is

$D = 16q^2 - 4(2q^2 - r) = 8q^2 + 4r$

$= 8\alpha^2\beta^2 + 4(\alpha^4 + \beta^4) = 4(\alpha^2 + \beta^2)^2$

> 0

Also, the product of the roots

$2q^2 - r = 2\alpha^2\beta^2 - (\alpha^4 + \beta^4) = -(\alpha^2 - \beta^2)^2$

< 0

$\Rightarrow$ one positive root and one negative root.

16. (c): The roots of the equation

$(1+m)x^2 - 2(1+3m)x + (1+8m) = 0$ where $m \in R - \{-1\}$

will be imaginary if $D < 0$

$\Rightarrow 4(1+3m)^2 - 4(1+m)(1+8m) < 0$

$\Rightarrow m(m-3) < 0 \Rightarrow m \in (0, 3)$

$\Rightarrow m = 1, 2$

17. (c): At least one root is negative *i.e.* one root is negative or both are negative.

When one root is negative, then $\alpha\beta < 0$

$\Rightarrow \dfrac{1+8m}{1+m} < 0 \Rightarrow (1+8m)(1+m) < 0$ or $1+m < 0$

$\Rightarrow m < -\dfrac{1}{8}$ or $m < -1$ *i.e.* $m \in \left(-1, -\dfrac{1}{8}\right)$ …(i)

When both the roots are negative then $\alpha + \beta < 0$

i.e. $\dfrac{2(1+3m)}{1+m} < 0 \Rightarrow 1+3m < 0$ and $1+m > 0$

$\Rightarrow m < \dfrac{-1}{3}$ and $m > -1 \Rightarrow m \in \left(-1, -\dfrac{1}{3}\right)$ …(ii)

Thus from (i) and (ii) $m \in \left(-1, -\dfrac{1}{8}\right)$ …(iii)

Since roots are real

$\therefore m(m-3) \geq 0$

$\Rightarrow m \in (-\infty, 0] \cup [3, \infty)$ …(iv)

Thus from (iii) and (iv), $m \in \left(-1, -\dfrac{1}{8}\right)$

18. (b) : Let the roots of the given equation be 2α and 3α

$\therefore 5\alpha = \dfrac{2(1+3m)}{1+m} \Rightarrow \alpha = \dfrac{2(1+3m)}{5(1+m)}$

and $6\alpha^2 = \dfrac{1+8m}{1+m}$

$\therefore 6 \cdot \dfrac{4}{25} \dfrac{(1+3m)^2}{(1+m)^2} = \dfrac{1+8m}{1+m}$

$\Rightarrow 25(8m+1)(m+1) = 24(3m+1)^2$ [$\because m \neq -1$]

$\Rightarrow 25(8m^2 + 9m + 1) = 24(9m^2 + 6m + 1)$

$\Rightarrow 16m^2 + 144m - 225m + 24 - 25 = 0$

$\Rightarrow 16m^2 - 81m - 1 = 0$ …(i)

$\because D = (-81)^2 - 4 \cdot 16(-1) > 0$

$\therefore$ The root of (i) are real and distinct.

i.e. the number of real values of m are 2.

19. (b) : Let $z = x + iy$

$\therefore \text{Im}(z) \geq 1 \Rightarrow y \geq 1$ … (i)

$\Rightarrow z - 2 - i = x + iy - 2 - i = (x-2) + (y-1)i$

$\Rightarrow |z - 2 - i| = 3 \Rightarrow (x-2)^2 + (y-1)^2 = 9$ … (ii)

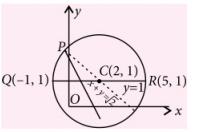
Now, $(1-i)z = (1-i)(x+iy) = (x+y) - i(x-y)$

Given, $\text{Re}((1-i)z) = \sqrt{2}$

$\Rightarrow x + y = \sqrt{2}$ … (iii)

Solving (i) and (ii), we get $Q(-1, 1)$ and $R(5, 1)$

$\therefore$ The set $S_1 \cap S_2 \cap S_3$ has one point namely the point P.



20. (c): From the above diagram, it is clear that $z = -1 + i$ and $z = 5 + i$ are the ends of a diameter of the circle $(x-2)^2 + (y-1)^2 = 3^2$

$\therefore PQ = |z - (-1 + i)|$; $PR = |z - (5 + i)|$

$\therefore PQ^2 + PR^2 = QR^2$

$\Rightarrow |z + 1 - i|^2 + |z - 5 - i|^2 = QR^2$

$= (5+1)^2 - (1-1)^2 = 36$

21. (d) : Any point ω satisfying $|\omega - 2 - i| < 3$ is a point in the circle with centre at C and radius 3 *i.e.* the circle in the figure.

$\therefore |z - \omega| < 6$

$|z - \omega| > ||z| - |\omega||$

$\therefore ||z| - |\omega|| < 6$

$\Rightarrow -6 < |z| - |\omega| < 6$

$\Rightarrow -6 + 3 < |z| - |\omega| + 3 < 6 + 3$

$\Rightarrow -3 < |z| - |\omega| + 3 < 9$

22. (A) - (s), (B) - (p), (C) - (q), (D) - (r)

(A) $\dfrac{a}{b} = \dfrac{6}{3} \Rightarrow \dfrac{2R\sin A}{2R\sin B} = 2$

$\left[\because \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R\right]$

$\Rightarrow \dfrac{\sin A}{\sin B} = \dfrac{2}{1} \Rightarrow \dfrac{\sin A + \sin B}{\sin A - \sin B} = \dfrac{3}{1}$

$\Rightarrow \dfrac{2\sin\left(\dfrac{A+B}{2}\right) \cdot \cos\left(\dfrac{A-B}{2}\right)}{2\cos\left(\dfrac{A+B}{2}\right) \cdot \sin\left(\dfrac{A-B}{2}\right)} = 3$

$\Rightarrow \tan\left(\dfrac{A+B}{2}\right) \cdot \cot\left(\dfrac{A-B}{2}\right) = 3$

$\Rightarrow \tan\left(\dfrac{\pi}{2} - \dfrac{C}{2}\right) = 3\tan\left(\dfrac{A-B}{2}\right)$ [$\because A + B + C = \pi$]

$\Rightarrow \cot\frac{C}{2}=3\times\frac{1}{3}=1$

$\left[\because \cos(A-B)=\frac{4}{5}.\ \therefore \tan\left(\frac{A-B}{2}\right)=\frac{1}{3}\right]$

$\Rightarrow \cot\frac{C}{2}=\cot\frac{\pi}{4} \Rightarrow C=\frac{\pi}{2}=90°$

$\therefore \Delta=\frac{1}{2}ab\sin C=\frac{1}{2}\times6\times3\times1=9$ sq. units

(B) $\frac{a}{\sin A}=\frac{b}{\sin B} \Rightarrow \frac{5}{\sin A}=\frac{7}{\sin B}$

$\Rightarrow \sin B=\frac{7}{5}\sin A=\frac{7}{5}\times\frac{3}{4}$

$\Rightarrow \sin B=\frac{21}{20}\ \left[\because \sin A=\frac{3}{4}\right]$

which is not possible.

(C) $\tan\frac{B}{2}\cdot\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\cdot\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

$=\sqrt{\frac{(s-a)^2}{s^2}}=\frac{s-a}{s}$

$\Rightarrow \cot\frac{B}{2}\cot\frac{C}{2}=\frac{s}{s-a}=\frac{a+b+c}{b+c-a}$

$=\frac{4a}{2a}=2\quad[\because b+c=3a]$

(D) $\frac{a(b^2+c^2)\cos A+b(c^2+a^2)\cos B+c(a^2+b^2)\cos C}{abc}$

$=\frac{1}{abc}\left[a(b^2+c^2)\cdot\frac{b^2+c^2-a^2}{2bc}+b(c^2+a^2)\frac{(c^2+a^2-b^2)}{2ac}\right.$

$\left.+c(a^2+b^2)\left(\frac{a^2+b^2-c^2}{2ab}\right)\right]$

$=\frac{1}{abc}\left[\frac{a^2(b^2+c^2)(b^2+c^2-a^2)+b^2(c^2+a^2)}{\frac{(c^2+a^2-b^2)+c^2(a^2+b^2)(a^2+b^2-c^2)}{2abc}}\right]$

$=\frac{1}{2(abc)^2}\left[\begin{matrix}a^2(b^2+c^2)^2+b^2(c^2+a^2)^2+c^2(a^2+b^2)^2\\-a^4(b^2+c^2)-b^4(c^2+a^2)-c^4(a^2+b^2)\end{matrix}\right]$

$=\frac{1}{2(abc)^2}[2a^2b^2c^2+2b^2c^2a^2+2c^2a^2b^2]$

$=\frac{1}{2(abc)^2}\times6(abc)^2=3$

23. (A) - (r), (B) - (p), (C) - (s), (D) - (s)

(A) $\tan^{-1}([\pi])+\tan^{-1}([-\pi]+1)$

$=\tan^{-1}3+\tan^{-1}(-4+1)$

$=\tan^{-1}3+\tan^{-1}(-3)=\tan^{-1}3-\tan^{-1}3=0$

$(\because \tan^{-1}(-x)=-\tan^{-1}x)$

(B) The given equation is $\tan x+\sec x=2\cos x$

$\Rightarrow 1+\sin x=2\cos^2 x$

$\Rightarrow (1+\sin x)=2(1+\sin x)(1-\sin x)$

$\Rightarrow (1+\sin x)(2\sin x-1)=0$

$\therefore \sin x=-1$ and $\sin x=1/2$

$\because \sin x\neq-1$, hence $\sin x=1/2\ [\because \cos x\neq0]$

$\Rightarrow x=\frac{\pi}{6},\ \frac{5\pi}{6}$

(C) $x+2\tan x=\frac{\pi}{2} \Rightarrow 2\tan x=\frac{\pi}{2}-x$

$\Rightarrow \tan x=\frac{\pi}{4}-\frac{x}{2}=y$ (say)

$\Rightarrow y=\tan x$ and $y=\frac{\pi}{4}-\frac{x}{2}$

(D) $x^3+x^2+4x+2\sin x=0$

$\Rightarrow x^3+(x+2)^2+2\sin x=4$

$x=0$ satisfies the equation

Now, in $0<x\leq\pi$, we have

$x^3+(x+2)^2+2\sin x>4$

and in $\pi<x\leq2\pi$, we have

$x^3+(x+2)^2+2\sin x>27+25-2$

Hence $x=0$ is the only solution.

24. (A) - (p, q) ; (B) - (r) ; (C) - (s) ; (D) - (s)

(A) $0\leq\sin^2\theta\leq\frac{1}{2},\ 0\leq\sin\theta\leq\frac{1}{2}$

(B) $\cot^2(x+y)+\tan^2(x+y)+y^2+2y-1=0$

$\Rightarrow \{\cot(x+y)-\tan(x+y)\}^2$

$+2\cot(x+y)\tan(x+y)+y^2+2y-1=0$

$\Rightarrow \{\cot(x+y)-\tan(x+y)\}^2+y^2+2y+1=0$

$\Rightarrow \{\cot(x+y)-\tan(x+y)\}^2+(y+1)^2=0$

$\therefore \cot(x+y)-\tan(x+y)=0$

$\Rightarrow 1-\tan^2(x+y)=0$

$\Rightarrow \tan(x+y)=\pm1=\tan\left(\pm\frac{\pi}{4}\right) \Rightarrow x+y=\pm\frac{\pi}{4}$

Also, $y+1=0 \Rightarrow y=-1$

(C) (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)

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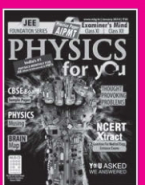

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